Self-consistent Many-Body Theory and Nuclear Matter in a Chiral Dirac-Hartree-Fock Approximation

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Abstract

A self-consistent chiral Dirac-Hartree-Fock (CDHF) approximation generated by an effective model of the (σ, ω, π) quantum hadrodynamics is discussed and applied to nuclear matter and neutron stars. The CDHF approximation maintains conditions of thermodynamic consistency connected to the fundamental requirement of Density Functional Theory (DFT). The self-consistent conditions to nuclear matter approximations generate functional equations for self-energies; accurate and rigorous solutions to self-energies are obtained and examined. The difference of solutions constructed by thermodynamic consistency (or DFT) and Feynman diagram approach is compared and discussed explicitly, which should be declared as an open question for many-body theory.

Exchange interactions are more important than direct interactions at nuclear matter saturation density, which suggests that an appropriate nuclear ground state approximation be the HF approximation rather than the mean-field (Hartree) approximation. The current CDHF approximation produces incompressibility and symmetry energy, $K = 218$ MeV and $a_4 = 21.3$ MeV. The application to neutron stars yields $M^{max}_{star}/M_{\odot} = 2.21$ in the unit of solar mass and radius $R = 11.6$ km, which improves mean-field results.

Keywords: Thermodynamic Consistency, Density Functional Theory, Feynman Diagram Approach and Self-energy Construction.

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1 Introduction

The relativistic mean-field models of Quantum Hadrodynamics (QHD) have provided a realistic description of bulk properties of finite nuclei, finite Fermi systems, nuclear matter and neutron stars [1–9], and they have been extended to effective models of the chiral $(\sigma, \pi, \omega, \rho)$ hadronic theories [10–13]. Historical motivations, successes and difficulties, evolutions and revolutions, reinterpretations and rebuttals are substantially discussed in chaps. 2 and 3 of the reference [13].

While the mean-field approximations to various nonlinear and chiral QHD models have been applied to properties of nuclear matter and neutron stars, the mean-field approximations introduced by replacing meson fields with classical fields are all equivalent to the Hartree approximation. The equivalence can be shown by self-consistent renormalization of physical quantities in terms of thermodynamic consistency [14,15], or by Landau's quasiparticle hypothesis [16,17] and the fundamental requirement of self-consistency in Density Functional Theory (DFT) [18,19].

Because the Hartree and Fock-exchange interactions have the same order of magnitude in terms of coupling constants, the mean-field approximation had to be extended to Hartree-Fock approximation [20–24]. However, due to theoretical and numerical difficulties to determine physical quantities in pseudo-scalar pion models [25–31], vacuum (ultra-violet) contributions [32], the problem of self-consistency in the Hartree-Fock approximation is not sufficiently examined and investigated. The theoretical and computational problems are related to self-consistent single particle energies, retardation effects and truncation schemes of higher order interactions. Self-consistency demands renormalization of physical quantities and requires the effective masses of nucleons and mesons, $M^*, m^*_\sigma, m^*_\pi, m^*_\omega$, effective coupling constants, $g^*_\sigma, g^*_\pi, g^*_\omega$, and redefinition of sources for equations of motion. It can be proven that nonlinear mean-field approximations are equivalent to Hartree approximations with correctly renormalized effective masses and effective coupling onstants [7, 8, 14, 15]. The self-consistent quasiparticle formalism produces density-dependent relations among physical quantities at saturation. In the current paper, the empirical value of binding energy at saturation is taken as -15.75 MeV at $k_F = 1.30$ fm^{-1} (at baryon density $\rho_B = 0.148$ fm^{-3}).

Historically, self-consistency has been discussed as the Hugenholtz-Van Hove (HV) theorem [33], which requires the equality between the Fermi energy and the average energy of a physical system at the density of zero pressure (at nuclear matter saturation point). If one fails to maintain the equality, the nuclear matter saturation properties will become ambiguous, and it is not possible to draw reliable conclusions for the nuclear matter calculations [34]. The HV theorem is generalized to the property of *conserving approximations* [35, 36], *thermodynamic consistency* [14, 15] with Landau' requirement of quasiparticles and equivalently, as the fundamental requirement of DFT [18, 19]. At present, the quantum many-body systems in equilibrium and non-equilibrium systems are active fields of research [37–40].

The self-consistency can be stated explicitly as follows: by starting from a Lagrangian or a Hamiltonian, one can derive the energy density, $\mathcal{E}$, and hydrodynamic pressure, P_{hy}, from energy-momentum tensor. One can also obtain the thermodynamic pressure P_T by applying the thermodynamic relation to the energy density: $P_T = \rho_B^2 \dfrac{\partial (\mathcal{E}/\rho_B)}{\partial \rho_B}$. Moreover, one can define the HV pressure: $P_{HV} = \rho_B E(k_F) - \mathcal{E}$, according to the HV theorem and Gibbs relation in thermodynamics, where $E(k_F)$ is the single particle energy derived from an approximation. Therefore, consistency conditions will require that approximations have to maintain the equality of pressures $P_{hy} = P_T = P_{HV}$ and the minimum property of binding energy at saturation density of nuclear matter, $P_{hy} = P_T = P_{HV} = 0$, simultaneously ($P_{hy} = P_T$ is called the virial theorem [41]). This is the fundamental constraint for nuclear matter approximations. Hence, before calculating any experimental properties of nuclear physics, one must discuss and show the fundamental constraints whether one's approximation maintains the above conditions.

The consistency is also examined as the equality of single particle energy defined in the poles of Green's function, $E_G(k_F)$, and the quasiparticle energy defined by Landau's hypothesis, $\delta\mathcal{E}/\delta n_i = E_L(k_F)$. The relation, $E_G(k_F) = E_L(k_F)$, should be at least approximately controlled in all calculations to extract reliable conclusions, which is not simply assumed equal at any density [1, 7, 8, 34]. When the equality of single particle energies is satisfied, thermodynamic relations and dynamical calculations become compatible. The mean-filed approximations in effective models of QHD satisfy the requirement accurately.

Although the relativistic Hartree approximations to the QHD maintain thermodynamic consistency exactly, it is difficult to maintain self-consistency and thermodynamic consistency in sophisticated approximations [42] (Hartree-Fock, Ring, *etc.*), since inclusion of certain interaction processes by physical intuitions as Brueckner and others proposed [43–46] will not necessarily promise theoretical consistency in the non-perturbative many-body and strongly interacting systems. Thermodynamic consistency, or equivalently $E_G(k_F) = E_L(k_F)$, is essential for many-body calculations.

Based on the chiral (σ, π, ω) mean-field approximation [8], the current Chiral Dirac-Hartree-Fock (CDHF) approximation is constructed with positive energy baryons in Fermi-sea particles, while negative energy particles in vacuum are neglected. Vacuum fluctuation corrections are performed in the level of mean-field approximations [1, 8], and corrections produced by counter-terms for ultra-violet divergences become finite and strictly density-dependent with the constraint at nuclear matter saturation. The vacuum fluctuation corrections are discussed in Hartree-Fock approximation, however, quantitative numerical values of the vacuum fluctuation corrections have not been examined yet [32], and the difficulties of renormalization persists in CDHF and other sophisticated approximations. However, it could be physically expected that the constraints at nuclear matter saturation strictly restrict high energy contributions to physical quantities, though the example is demonstrated only in linear and nonlinear Hartree approximations.

We will show that Hartree-Fock approximation is a physically reasonable ground state approximation for nuclear matter than mean-field approzimations. This fact is shown explicitly by comparing Hartree and Fock contributions respectively in sec. 4. Hence, mean-field

(Hartree) analyses performed by many researchers are not appropriate. Thermodynamic consistency, equivalently $E_G(k_F) = E_L(k_F)$, is accurately satisfied with the renormalized CDHF calculations, and the theoretical relations among single particle energy, self-consistency and retardation mechanism are discussed. In the course of analysis, the discrepancy between Feynman diagram approach and the functional differentiation to determine self-energies is examined. This approach may help construct thermodynamically consistent approximations for analyzing non-perturbative calculations for nuclear matter.

The CDHF approximation and self-consistency, retardation corrections, discrepancy between Feynman diagram and DFT approach are explained in sec. 3. The numerical results of HF approximation, different saturation properties of direct and exchange corrections, physical quantities at saturation are discussed in sec. 4. Properties of symmetric nuclear matter and neutron stars are discussed in sec. 5, and conclusions are in sec. 6.

2 The chiral mean-field Lagrangian, $\mathcal{L}_{csb}$, for CDHF

The notations and signs are introduced to comply with QHD models. The chiral $(\sigma, \boldsymbol{\pi}, \omega)$ mean-field approximation is defined by replacing meson quantum fields with classical fields: $\hat{\sigma} \to \langle\sigma\rangle = \sigma$ and $\hat{\omega}_\mu = (\hat{\omega}_0, \hat{\boldsymbol{\omega}}) \to (\omega_0, \boldsymbol{\omega})$. The spatial part of the vector field $\langle\boldsymbol{\omega}\rangle$ should vanish by the requirement of rotational invariance in the (mean-field) Hartree approximation, while contributions of the vector component of ω-meson, $\boldsymbol{\omega}$, and pion field, $\boldsymbol{\pi}$, are restored on the level of HF approximation. It should be noted that we employed the chiral-invariant potential,

$$V(\sigma, \pi, \omega_\mu) = \frac{\lambda}{4}\Big\{(\sigma^2+\pi^2)(\sigma^2+\pi^2-a\omega_\mu^2) - a^2(\omega_\mu\omega^\mu)^2\Big\}, \qquad (\lambda > 0)\ , \tag{2.1}$$

in order to produce nuclear ground state by symmetry breaking mechanism. The new ground state is defined by $\hat{\sigma} \to \langle\sigma\rangle + \phi$ and $\langle\sigma\rangle = -M/g$, $\langle\omega\rangle = 0$ and $\langle\pi\rangle = 0$ [8, 10, 13], and the notation of scalar meson field is changed to ϕ (σ is used for index notations, such as m_σ, g_σ,...).

The chiral mean-field Lagrangian is generated after symmetry-breaking, resulting in:

$$\begin{aligned}
\mathcal{L}_{csb} = {} & \sum_n \bar{\psi}_n\Big[\gamma^\mu(i\partial_\mu - g_\omega\omega_\mu) - \big\{M_n - g(\phi+\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi})\big\}\Big]\psi_n \\
& + \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m_\sigma^2\phi^2) + \frac{m_\sigma^2-m_\pi^2}{2}\frac{g}{M}\{\phi - \frac{g}{4M}(\phi^2+\boldsymbol{\pi}^2-a\omega_\mu^2)\}\phi^2 \\
& + \frac{1}{2}(\partial_\mu\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi} - m_\pi^2\boldsymbol{\pi}^2) + \frac{m_\sigma^2-m_\pi^2}{2}\frac{g}{M}\{\phi - \frac{g}{4M}(\phi^2+\boldsymbol{\pi}^2-a\omega_\mu^2)\}\pi^2 \\
& - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_\omega^2\omega_\mu^2 - \frac{m_\sigma^2-m_\pi^2}{2}\frac{g}{M}\{\frac{1}{2}a\phi - \frac{g}{4M}a^2\omega_\mu^2\}\omega_\mu^2\ ,
\end{aligned} \tag{2.2}$$

where $a = 2m_\omega^2/m_\pi^2$, and $g_\sigma = g_\pi$ is required from invariance under the chiral transformation and denoted as g; the field strength $F^{\mu\nu}$ is written as $F_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu$. The hadron masses are fixed as $M = 939$, $m_\omega = 783.0$ and $m_\pi = 138.0$ MeV. The model has three parameters to produce the nuclear matter saturation property: binding energy, $\mathcal{E}/\rho_B - M = -15.75$ MeV, at baryon density, $\rho_B = 0.148$ fm^{-3}. In the effective chiral symmetry breaking model, they are the mass of neutral scalar meson, m_σ, and coupling constants g and g_ω.

However, the ratios of physical parameters, $g_\sigma^2(M/m_\sigma)^2$, $g_\omega^2(M/m_\omega)^2$, are known to contribute to physical quantities in QHD models, and these parameters respectively take almost constant values [1]. The current CDHF calculation is compatible with the ratios of physical parameters and analyses in QHD, and the characteristic constants, $g^2(M/m_\sigma)^2$ and $g_\omega^2(M/m_\omega)^2$, are constrained by the saturation condition, $\mathcal{E}/\rho_B - M = -15.75$ MeV, at $\rho_B = 0.148$ fm^{-3} because of self-consistency. The values of constants are, $g^2(M/m_\sigma)^2 = 222$ and $g_\omega^2(M/m_\omega)^2 = 109$, and they are fixed as $m_\sigma = 70$ MeV, $g = 1.110$, $g_\omega = 8.712$, resulting in improved data, (see, Table 1). The mean-field lagrangian of $\mathcal{L}_{csb}$ and induced mean-field (Hartree) approximation are discussed in references [7, 8, 13–15].

The baryons (neutrons and protons) are described by the Dirac equation of motion and Schwinger-Dyson equation is used to sum all orders of self-consistent Feynman diagrams to the baryon Green's function. The baryon propagator of the relativistic approximations is constructed analogously to the noninteracting propagator, which is essential for the quasiparticle approach [47]. It is assumed that the baryon Green's function has simple poles with unit residues and at a finite baryon density the particles are filled up to the Fermi surface, $|\boldsymbol{k}| = k_F$.

The Green's function is defined in the rest frame of nuclear matter in terms of the particle-antiparticle propagator, $G_F(k)$, and the hole propagator inside the Fermi-sea, $G_D(k)$, as follows:

$$\begin{aligned} G(k) &= G_F(k) + G_D(k) \ , \\ G_F(k) &= (\gamma^\mu k_\mu^* + M^*(k)) \frac{1}{k_\mu^{*2} - M^*(k)^2 + i\epsilon} \ , \\ G_D(k) &= (\gamma^\mu k_\mu^* + M^*(k)) \frac{i\pi}{E^*(k)} \delta(k^0 - E(\boldsymbol{k}))\theta(k_F - |\boldsymbol{k}|) \ , \end{aligned} \tag{2.3}$$

where $E(k)$ is the self-consistent single particle energy spectrum, and $\epsilon = 0^+$, a positive infinitesimal number. The self-consistent dynamical variables (all are functions of $|\boldsymbol{k}|, k^0$ and k_F) are defined as:

$$\begin{aligned} M^*(\boldsymbol{k}) \equiv M + \Sigma^s(\boldsymbol{k}) \ , \qquad \boldsymbol{k}^*(\boldsymbol{k}) &\equiv \boldsymbol{k}(1 + \Sigma^v(\boldsymbol{k})) \ , \\ k^{*0} \equiv E^*(\boldsymbol{k}) &\equiv (\boldsymbol{k}^{*2}(k) + M^*(\boldsymbol{k})^2)^{1/2} \ , \\ k^{*\mu} \equiv k^\mu + \Sigma^\mu(\boldsymbol{k}) &= (k^0 + \Sigma^0(\boldsymbol{k}), \boldsymbol{k}^*(k)) \ . \end{aligned} \tag{2.4}$$

The single particle energy is given by the solution to the transcendental equation,

$$\begin{aligned} E(\boldsymbol{k}) &= [E^*(\boldsymbol{k}) - \Sigma^0(\boldsymbol{k})]_{k_0 = E(\boldsymbol{k})} \\ &= \{k^2[1 + \Sigma^v(|\boldsymbol{k}|, E(\boldsymbol{k}))]^2 + [M + \Sigma^s(|\boldsymbol{k}|, E(\boldsymbol{k}))]^2\}^{1/2} - \Sigma^0(|\boldsymbol{k}|, E(\boldsymbol{k})) \ , \end{aligned} \tag{2.5}$$

and self-energies depend on $E(\boldsymbol{k}), |\boldsymbol{k}|$ and k_F. The Green's functions with the dynamical variables, direct and exchange Feynman diagrams and the energy-momentum tensor defined by lagrangian after symmetry breaking are used to obtain the ground-state energy density, $\mathcal{E}$, and the hydrodynamic pressure, P_{hy} [7, 8, 10–13].

The self-energies, $\Sigma^s(\boldsymbol{k})$, $\Sigma^0(\boldsymbol{k})$, $\Sigma^v(\boldsymbol{k})$ are constructed field-theoretically by including direct and exchange terms using Schwinger-Dyson formalism, but negative energy state contributions are excluded. The energy density and pressure are derived from the ground state expectation value of the energy-momentum tensor. The dynamical variables, such as $M^*(\boldsymbol{k})$, $\boldsymbol{k}^*(\boldsymbol{k})$ and $E(\boldsymbol{k})$ (or self-energies $\Sigma^s(\boldsymbol{k})$, $\Sigma^v(\boldsymbol{k})$ and $\Sigma^0(\boldsymbol{k})$), are determined self-consistently in sec. 3.

3 The energy density in the chiral Dirac-Hartree-Fock approximation

The energy density in the chiral Dirac-Hartree-Fock (CDHF) approximation is expressed as:

$$\mathcal{E}_{HF} = \mathcal{E}_B + \mathcal{E}_H(\phi, \omega) + \mathcal{E}_{EX}(\phi, \omega, \pi) \tag{3.1}$$

where $\mathcal{E}_B(k_F)$, $\mathcal{E}_H(\phi, \omega)$ and $\mathcal{E}_{EX}(\phi, \omega, \pi)$ are the baryon, direct (Hartree) and exchange energy densities, respectively.

The baryon energy density is given by the self-consistent single particle energy of nucleons:

$$\mathcal{E}_B(k_F) = \sum_i n_i E(\boldsymbol{k}_i) = \sum_{B=n,p} \frac{2}{(2\pi)^3} \int^{k_{F_B}} d^3k E_B(\boldsymbol{k}), \tag{3.2}$$

where n_i is the particle occupation number, and the baryon density is denoted as:

$$\rho_B = \sum_i n_i = \frac{\zeta}{6\pi^2} k_F^3 \ , \tag{3.3}$$

where ζ is the spin-isospin degeneracy factor, $\zeta = 2$ (neutron matter), $\zeta = 4$ (nuclear matter), and k_{F_B} is a baryon Fermi-momentum ($B = n, p$).

The Hartree energy density, $\mathcal{E}_H(\phi, \omega)$, is,

$$\begin{aligned} \mathcal{E}_H(\phi, \omega) &= \frac{1}{2} m_\sigma^2 \phi^2 - \frac{g}{2M}(m_\sigma^2 - m_\pi^2)\left(\phi - \frac{1}{2}\frac{g}{2M}\phi^2\right)\phi^2 \\ &- \frac{1}{2} m_\omega^2 \omega_0^2 + \frac{g}{2M}(m_\sigma^2 - m_\pi^2) a \left(\phi + \frac{1}{2}\frac{g}{2M} a\omega_0^2 - \frac{g}{2M}\phi^2\right)\omega_0^2, \end{aligned} \tag{3.4}$$

where the constant, $a = 2m_\omega^2/m_\pi^2$, is required in the new nuclear ground state due to symmetry-breaking mechanism. One should note that meson fields of $\mathcal{E}_H(\phi, \omega)$ are only density-dependent by way of Fermi-momentum, k_F, as $\phi(k_F)$ and $\omega_0(k_F)$.

The exchange energy, $\mathcal{E}_{EX}(\phi,\omega,\pi)$, is:

$$\mathcal{E}_{EX}(\phi,\omega,\pi) = \frac{1}{2\zeta}\sum_i\sum_j n_i n_j \frac{1}{E^*(\boldsymbol{k}_i)E^*(\boldsymbol{q}_j)}\times$$
$$\Big\{g_\sigma^2 D_\sigma^0(k_i-q_j)\left[\frac{1}{2}-\{V_\sigma(\boldsymbol{k}_i-\boldsymbol{q}_j)+[E(\boldsymbol{k}_i)-E(\boldsymbol{q}_j)]^2\}D_\sigma^0(k_i-q_j)\right]\left(k_i^{*\mu}q_{j\mu}^*+M^*(\boldsymbol{k}_i)M^*(\boldsymbol{q}_j)\right)$$
$$+2g_\omega^2 D_\omega^0(k_i-q_j)\left[\frac{1}{2}-\left\{V_\omega(\boldsymbol{k}_i-\boldsymbol{q}_j)+[E(\boldsymbol{k}_i)-E(\boldsymbol{q}_j)]^2\right\}D_\omega^0(k_i-q_j)\right]\left(k_i^{*\mu}q_{j\mu}^*-2M^*(\boldsymbol{k}_i)M^*(\boldsymbol{q}_j)\right)$$
$$-(\zeta-1)g_\pi^2 D_\pi^0(k_i-q_j)\left[\frac{1}{2}-\left\{V_\pi(\boldsymbol{k}_i-\boldsymbol{q}_j)+[E(\boldsymbol{k}_i)-E(\boldsymbol{q}_j)]^2\right\}D_\pi^0(k_i-q_j)\right]\cdot$$
$$\left(-k_i^{*\mu}q_{j\mu}^*+M^*(\boldsymbol{k}_i)M^*(\boldsymbol{q}_j)\right)\Big\}\ . \tag{3.5}$$

One should note that the meson propagators in Eq. (3.5) are self-consistently determined as,

$$D_i^0(k)=(k_0^2-\boldsymbol{k}^2-m_i^{*2}(\boldsymbol{k}))^{-1}, \qquad (i=\sigma,\omega,\pi)\ . \tag{3.6}$$

The coupling constants, $g_\sigma = g_\pi \equiv g$, should be understood from chiral symmetry-breaking mechanism. One should note that meson fields of $\mathcal{E}_{EX}(\phi,\omega,\pi)$ are Fermi-momentum and momentum dependent as $\phi(k_F,\boldsymbol{k})$, $\omega_\mu(k_F,\boldsymbol{k})$ and $\pi(k_F,\boldsymbol{k})$, and they are written as $\phi(\boldsymbol{k})$, $\omega_\mu(\boldsymbol{k})$ and $\pi(\boldsymbol{k})$ for simplicity.

The nonlinear interaction modifications, $V_i(k_F,\boldsymbol{k})$ (denoted as $V_i(\boldsymbol{k})$, $i=\sigma,\omega,\pi$), to energy transfer $[E(\boldsymbol{k}_i)-E(\boldsymbol{q}_j)]^2$ are given by:

$$\begin{aligned}
V_\sigma(\boldsymbol{k}) &= \frac{m_\sigma^2-m_\sigma^{*2}(\boldsymbol{k})}{2}-\frac{m_\sigma^2-m_\pi^2}{2}\frac{g}{M}\Big\{\phi(\boldsymbol{k})-\frac{1}{4}\frac{g}{M}(\phi^2(\boldsymbol{k})+\pi^2(\boldsymbol{k})-a\omega_\mu^2(\boldsymbol{k}))\Big\}\\
V_\omega(\boldsymbol{k}) &= \frac{m_\omega^2-m_\omega^{*2}(\boldsymbol{k})}{2}-\frac{m_\sigma^2-m_\pi^2}{2}\frac{ga}{M}\Big\{\frac{\phi(\boldsymbol{k})}{2}-\frac{1}{4}\frac{g}{M}a\omega_\mu^2(\boldsymbol{k})\Big\}\\
V_\pi(\boldsymbol{k}) &= \frac{m_\pi^2-m_\pi^{*2}(\boldsymbol{k})}{2}-\frac{m_\sigma^2-m_\pi^2}{2}\frac{g}{M}\Big\{\phi(\boldsymbol{k})-\frac{1}{4}\frac{g}{M}(\phi^2(\boldsymbol{k})+\pi^2(\boldsymbol{k})-a\omega_\mu^2(\boldsymbol{k}))\Big\}\ .
\end{aligned} \tag{3.7}$$

It is noticeable that nonlinear meson corrections in $\mathcal{E}_H(\phi,\omega)$ and $\mathcal{E}_{EX}(\phi,\omega,\pi)$ have a characteristic form of coefficients:

$$\frac{m_\sigma^2-m_\pi^2}{2}\frac{g}{M}\ . \tag{3.8}$$

The complex nonlinear corrections seem to be suppressed by $g/M \ll 1$, but plus or minus sign is determined by the mass difference, $m_\sigma^2-m_\pi^2$. When $m_\sigma > m_\pi$, exchange contributions of self-energies become large, and solutions cannot be obtained as $m_\sigma \gtrsim 300$. The similar problem was pointed out in other calculations of pseudo-scalar pion model, resulting in the extension to pseudo-vector pion model. The problem is known as sensitive cancellations of large pion self-energies at low-energy densities [25–29]. The CDHF approximation produced by the current effective pseudo-scalar pion model exhibits the similar problem when $m_\sigma > m_\pi$, however, when $m_\sigma < m_\pi$, finite and improved results are obtained, which suggests that self-consistency might be essential for sensitive cancellations.

By performing the first variation of $\mathcal{E}_{HF}$ with respect to n_i, one can produce the equation for single particle energy and self-energies as:

$$\frac{\delta\mathcal{E}_{HF}}{\delta n_i} = E(\boldsymbol{k}_i) + \sum_l \left[\frac{\delta M^*(\boldsymbol{k}_l)}{\delta n_i} \frac{\delta\mathcal{E}}{\delta M^*(\boldsymbol{k}_l)} + \frac{\delta\boldsymbol{k}^*(k_l)}{\delta n_i} \cdot \frac{\delta\mathcal{E}}{\delta\boldsymbol{k}^*(\boldsymbol{k}_l)} + \frac{\delta\Sigma^0(\boldsymbol{k}_l)}{\delta n_i} \frac{\delta\mathcal{E}}{\delta\Sigma^0(\boldsymbol{k}_l)} \right] , \quad (3.9)$$

and by requiring the terms in the functional differential form in the right-hand side equal to 0, the self-consistent single particle energy, $E(\boldsymbol{k}_i)$, is rigorously obtained, and coupled functional integro-differential equations for self-energies are generated as [21, 22, 24]:

$$\frac{\delta M^*(\boldsymbol{k}_l)}{\delta n_i} \frac{\delta\mathcal{E}}{\delta M^*(\boldsymbol{k}_l)} = 0, \quad \frac{\delta\boldsymbol{k}^*(\boldsymbol{k}_l)}{\delta n_i} \cdot \frac{\delta\mathcal{E}}{\delta\boldsymbol{k}^*(\boldsymbol{k}_l)} = 0, \quad \frac{\delta\Sigma^0(\boldsymbol{k}_l)}{\delta n_i} \frac{\delta\mathcal{E}}{\delta\Sigma^0(\boldsymbol{k}_l)} = 0 \ . \quad (3.10)$$

The equations (3.9) and (3.10) are the requirement of thermodynamic consistency. Therefore, *thermodynamic consistency* reproduces the fundamental requirement of DFT in terms of self-energies. It is possible to directly obtain exact or accurate solutions to the coupled functional equations for self-energies from (3.10). One should note that the self-energy solutons can be compared to those derived from Feynman diagram method. It can be shown explictly that self-energies derived from thermodynamic consistency (3.10) and Feynman diagram method are essentially different in evaluating overall energy interactions (retardation interactions) and nonlinear interactions [21–24].

The effective masses of mesons are derived from equations of motions and self-consistent conditions, (3.9) and (3.10), and they are given by:

$$\begin{aligned}
m_\sigma^{*2} &= m_\sigma^2 - \frac{m_\sigma^2 - m_\pi^2}{2}\frac{g}{M}3\phi(\boldsymbol{k}) + \frac{m_\sigma^2 - m_\pi^2}{2}(\frac{g}{M})^2\Big(\phi^2(\boldsymbol{k}) + \pi^2(\boldsymbol{k}) - \frac{a}{2}\omega_\mu^2(\boldsymbol{k})\Big) \ , \\
m_\omega^{*2} &= m_\omega^2 - \frac{m_\sigma^2 - m_\pi^2}{2}\frac{g}{M}a\phi(\boldsymbol{k}) + \frac{m_\sigma^2 - m_\pi^2}{2}(\frac{g}{M})^2\Big[\frac{1}{2}(\phi^2(\boldsymbol{k}) + \pi^2(\boldsymbol{k})) + a\omega_\mu^2(\boldsymbol{k})\Big]a \ , \quad (3.11) \\
m_\pi^{*2} &= m_\pi^2 - \frac{m_\sigma^2 - m_\pi^2}{2}\frac{g}{M}2\phi(\boldsymbol{k}) + \frac{m_\sigma^2 - m_\pi^2}{2}(\frac{g}{M})^2\Big(\phi^2(\boldsymbol{k}) + \pi^2(\boldsymbol{k}) - \frac{1}{2}a\omega_\mu^2(\boldsymbol{k})\Big) \ .
\end{aligned}$$

It should be stressed that effective masses of mesons and nucleons are required in eqs. (3.2) $\sim$ (3.6) explicitly because of self-consistency. The effective masses of mesons are density-dependent, $m_i^*(k_F)$ in the Hartree approximation, but they are density and momentum dependent $m_i^*(k_F, \boldsymbol{k})$ in HF approximation. Meson effective masses $m_i^*(\boldsymbol{k})$, the effective mass of nucleons $M^*(\boldsymbol{k})$, single particle energy, $E(\boldsymbol{k})$ and momentum $k^*(\boldsymbol{k})$, are self-consistently determined. These physical quantities are fundamentally related to binding energies at saturation of nuclear matter.

Let us denote the HF self-energies calculated from Feynman diagrams as $\Sigma_F(\boldsymbol{k})$, and self-energies calculated directly from (3.10) as $\Sigma_L(\boldsymbol{k})$. It is explicitly discussed in [22–24] that self-energies constructed from Feynman diagram method, $\Sigma_F(k)$, are approximate solutions to (3.9) and (3.10). The self-energies are equivalent, $\Sigma_F(\boldsymbol{k}) = \Sigma_L(\boldsymbol{k})$ in mean-field (Hartree) approximations and in the static limit of DHF approximation [21].

The static limit is defined as energy-transfers should vanish, $E(\boldsymbol{k}) - E(\boldsymbol{q}) \to 0$, in all calculations of the HF calculation. The equality, $\Sigma_F(\boldsymbol{k}) = \Sigma_L(\boldsymbol{k})$, is proved by directly performing the

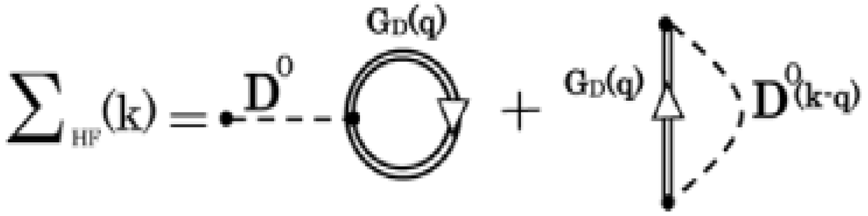

Fig. 1 The Hartree-Fock self-energy drawn by propagators of baryons, $G_D(\boldsymbol{k})$, and mesons, $D_i^0(\boldsymbol{k})$. Meson interaction lines, $D_i^0(\boldsymbol{k})$ $(i = \sigma, \pi, \omega)$, are given by effective masses of mesons, self-consistently renormalized as (3.6) and (3.11).

functional derivative (3.10) and comparing the deduced $\Sigma_L(\boldsymbol{k})$ with $\Sigma_F(\boldsymbol{k})$ constructed from Feynman diagrams, respectively. The fact indicated that the difference between $\Sigma_F(\boldsymbol{k})$ and $\Sigma_L(\boldsymbol{k})$ originates from energy transfer and retardation interactions. The self-energy solutions, $\Sigma_L(\boldsymbol{k})$, include retardation corrections accurately to maintain (3.9) and (3.10). The residual density-dependent, retardation terms cause inconsistency, but the numerical violation is very small since $P_{hy} = P_T = P_{HV}$ is controlled accurately in the self-consistent CDHF approximation. One should note that retardation corrections can not be correctly produced in $\Sigma_F(\boldsymbol{k})$ by way of Feynman diagram constructions.

The functional derivatives leave residual density-dependent terms related to three-body interactions, which should be connected to two-body Green's function $G(k_1, k_2)$. Hence, residual terms could be produced in hierarchical structure connected to higher order Green's functions [48–50]. The higher order contributions are negligible as long as (3.9) and (3.10) are satisfied, which are checked by calculating pressures: $P_{hy} = P_T = P_{HV}$, and it may help understand sensitive cancellations of residual corrections in self-energies.

Therefore, the conditions of thermodynamic consistency suggest that accurate, consistent solutions be possible to construct by renormalizing 3-body and higher-order interactions. Thermodynamic consistency is a method to renormalize and decouple higher order corrections, but it is different from the artificial decoupling of equations of motion in hierarchical, infinite BBGKY chain problems [49, 50].

We denote the accurate self-energies derived from functional equations (3.9) $\sim$ (3.10) as *admissible solutions* to self-consistent approximations. They produce the effective mass of nucleons:

$$
\begin{aligned}
M^*(\boldsymbol{k}_i) &= M - \frac{g_\sigma^2}{m_\sigma^2}\rho'_\sigma \\
-\frac{1}{\zeta}\sum_l n_l \frac{M^*(\boldsymbol{q}_l)}{E^*(\boldsymbol{q}_l)} &\Big\{ g_\sigma^2 D_\sigma^0(k_i - q_l)\Big[\tfrac{1}{2} - \{V_\sigma(\boldsymbol{k}_i - \boldsymbol{q}_l) + (E(\boldsymbol{k}_i) - E(\boldsymbol{q}_l))^2\} D_\sigma^0(k_i - q_l)\Big] \\
&- 4g_\omega^2 D_\omega^0(k_i - q_l)\Big[\tfrac{1}{2} - \{V_\omega(\boldsymbol{k}_i - \boldsymbol{q}_l) + (E(\boldsymbol{k}_i) - E(\boldsymbol{q}_l))^2\} D_\omega^0(k_i - q_l)\Big] \\
&- (\zeta - 1) g_\pi^2 D_\pi^0(k_i - q_l)\Big[\tfrac{1}{2} - \{V_\pi(\boldsymbol{k}_i - \boldsymbol{q}_l) + (E(\boldsymbol{k}_i) - E(\boldsymbol{q}_l))^2\} D_\pi^0(k_i - q_l)\Big]\Big\},
\end{aligned}
\tag{3.12}
$$

where $M^*(\boldsymbol{k}_i) \longrightarrow M, (k_F \to 0)$ is used. The scalar density, ρ'_σ, is defined as,

$$\rho'_\sigma(k_F) = \frac{g_\sigma^2}{m_\sigma^{*2}} \Big\{ \sum_i n_i \frac{M^*(\boldsymbol{k}_i)}{E^*(\boldsymbol{k}_i)} - \frac{m_\sigma^2 - m_\pi^2}{4M} a\omega_0^2 \Big\} \; . \tag{3.13}$$

The modified momentum, $\boldsymbol{k}^*$, is,

$$\begin{aligned} \boldsymbol{k}^*(\boldsymbol{k}_i) = \boldsymbol{k}_i &+ \frac{1}{\zeta} \frac{\boldsymbol{k}_i}{|\boldsymbol{k}_i|} \cdot \sum_l n_l \frac{\boldsymbol{q}^*(\boldsymbol{q}_l)}{E^*(\boldsymbol{q}_l)} \cdot \\ &\Big\{ g_\sigma^2 D_\sigma^0(k_i - q_l) \Big[\tfrac{1}{2} - \{ V_\sigma(\boldsymbol{k}_i - \boldsymbol{q}_l) + (E(\boldsymbol{k}_i) - E(\boldsymbol{q}_l))^2 \} D_\sigma^0(k_i - q_l) \Big] \\ &+ 2g_\omega^2 D_\omega^0(k_i - q_l) \Big[\tfrac{1}{2} - \{ V_\omega(\boldsymbol{k}_i - \boldsymbol{q}_l) + (E(\boldsymbol{k}_i) - E(\boldsymbol{q}_l))^2 \} D_\omega^0(k_i - q_l) \Big] \\ &+ (\zeta - 1) g_\pi^2 D_\pi^0(k_i - q_l) \Big[\tfrac{1}{2} - \{ V_\pi(\boldsymbol{k}_i - \boldsymbol{q}_l) + (E(\boldsymbol{k}_i) - E(\boldsymbol{q}_l))^2 \} D_\pi^0(k_i - q_l) \Big] \Big\} \; , \end{aligned} \tag{3.14}$$

where $\boldsymbol{k}^*(\boldsymbol{k}_i) \longrightarrow \boldsymbol{k}_i$ $(k_F \to 0)$ is used. The admissible solution, $\Sigma^0(\boldsymbol{k})$, is:

$$\begin{aligned} \Sigma^0(\boldsymbol{k}_i) = &- \frac{g_\omega^2}{m_\omega^{*2}} \rho_B \\ &+ \frac{1}{\zeta} \sum_l n_l \Big\{ g_\sigma^2 D_\sigma^0(k_i - q_l) \Big[\tfrac{1}{2} - \{ V_\sigma(\boldsymbol{k}_i - \boldsymbol{q}_l) + (E(\boldsymbol{k}_i) - E(\boldsymbol{q}_l))^2 \} D_\sigma^0(k_i - q_l) \Big] \\ &+ 2g_\omega^2 D_\omega^0(k_i - q_l) \Big[\tfrac{1}{2} - \{ V_\omega(\boldsymbol{k}_i - \boldsymbol{q}_l) + (E(\boldsymbol{k}_i) - E(\boldsymbol{q}_l))^2 \} D_\omega^0(k_i - q_l) \Big] \\ &+ (\zeta - 1) g_\pi^2 D_\pi^0(k_i - q_l) \Big[\tfrac{1}{2} - \{ V_\pi(\boldsymbol{k}_i - \boldsymbol{q}_l) + (E(\boldsymbol{k}_i) - E(\boldsymbol{q}_l))^2 \} D_\pi^0(k_i - q_l) \Big] \Big\} \; . \end{aligned} \tag{3.15}$$

The self-energies are then related to dynamical variables and classical fields as:

$$\begin{aligned} M_{HF}^*(\boldsymbol{k}) &\equiv M + \Sigma_H^s(k_F) + \Sigma_F^s(\boldsymbol{k}) = M - g_\sigma(\phi_{HF}^D(k_F) + \phi_{HF}^{EX}(\boldsymbol{k})) \; , \\ \boldsymbol{k}_{HF}^*(\boldsymbol{k}) &\equiv \boldsymbol{k}(1 + \Sigma_F^v(\boldsymbol{k})) = \boldsymbol{k}(1 - g_\omega |\boldsymbol{\omega}_F(\boldsymbol{k})|) \; , \\ \Sigma_{HF}^0(\boldsymbol{k}) &= \Sigma_H^0(k_F) + \Sigma_F^0(\boldsymbol{k}) = -g_\omega(\omega_{HF}^{0D}(k_F) + \omega_{HF}^{0EX}(\boldsymbol{k})) \; , \\ k^{*0} &\equiv E_{HF}^*(\boldsymbol{k}) \equiv (\boldsymbol{k}_{HF}^{*2}(k) + M_{HF}^*(\boldsymbol{k})^2)^{1/2} \; , \\ k^{*\mu} &\equiv k^\mu + \Sigma_{HF}^\mu(\boldsymbol{k}) = (k^0 + \Sigma_{HF}^0(\boldsymbol{k}), \boldsymbol{k}_{HF}^*(k)) \; . \end{aligned} \tag{3.16}$$

The relations between self-energies and sigma fields are explicitly written in (3.16) as the k_F-dependent direct part, $\Sigma_H^s(k_F) = -g_\sigma \phi_{HF}^D(k_F)$, and density-momentum dependent exchange part, $\Sigma_F^s(\boldsymbol{k}) = -g_\sigma \phi_{HF}^{EX}(\boldsymbol{k})$, and similarly, $\Sigma_H^0(k_F) = -g_\omega \omega_{HF}^{0D}(k_F)$ and $\Sigma_F^0(\boldsymbol{k}) = -g_\omega \omega_{HF}^{0EX}(\boldsymbol{k})$ should be understood. This is important to make sure that the direct part, $\phi_{HF}^D(k_F)$ and $\omega_{HF}^{0D}(k_F)$ should satisfy the mean-field equations of motion for ϕ and ω. The total fields, $\phi_{HF}(\boldsymbol{k})$ and $\omega_{HF}^0(\boldsymbol{k})$ are directly connected to $M_{HF}^*(\boldsymbol{k})$ and $\Sigma_{HF}^0(\boldsymbol{k})$, respectively. When $M^*(\boldsymbol{k})$, $k^*(\boldsymbol{k})$ and $\Sigma^0(\boldsymbol{k})$ are derived from (3.12) $\sim$ (3.15), one must check again if the new scalar and omega fields, $\phi_{HF}^D(k_F)$ and $\omega_{HF}^{0D}(k_F)$, are the solution to mean-field equations of motion for mesons. The calculations converge reasonably well within $k_F = 2.0$ fm^{-1}, and the result will be applied to maximum masses of neutron stars [7, 8] in sec. 5.

One should note that the self-energies derived from Feynman diagrams, $\Sigma_F(k)$, do not agree with the results: (3.12) $\sim$ (3.15), and hence, $\Sigma_F(k)$ is not the self-consistent solution to (3.9) and (3.10). The energy-dependent terms in self-energies: $-\{V_i(\boldsymbol{k}-\boldsymbol{q})+(E(\boldsymbol{k})-E(\boldsymbol{q}))^2\}$, $(i=\sigma,\omega,\pi)$, are not produced from Feynman diagram approach [20]. These energy-dependent or retardation terms are produced generally from the energy-momentum tensor [1].

In the current chiral Dirac-Hartree-Fock (CDHF) approximation, thermodynamic consistency is accurate, whose accuracy is examined by numerically checking $P_{hy}=P_T=P_{HV}$, or $E_G(k_F)=E_L(k_F)$ (see the discussion in the sec. 1). Self-consistency is discussed in detail in the references [22, 24], and it is important to understand retardation and correlation effects for applications to nuclear physics and nuclear astrophysics.

4 The numerical results of CDHF approximation

The self-energies (3.12) $\sim$ (3.15) are coupled nonlinear integral equations to determine $M^*(\boldsymbol{k}), \boldsymbol{k}^*(\boldsymbol{k})$ and $E(\boldsymbol{k})$ (or $\Sigma^0(\boldsymbol{k})$), and they are solved by iterations with starting values of Hartree solutions [21, 22, 24]. The momentum-dependent nonlinear corrections in self-energies and energy density, $V_i(\boldsymbol{k}-\boldsymbol{q})$, or alternatively, $\phi(\boldsymbol{k}-\boldsymbol{q})$, $\omega_\mu(\boldsymbol{k}-\boldsymbol{q})$, $\pi(\boldsymbol{k}-\boldsymbol{q})$, and meson effective masses $m_i^*(\boldsymbol{k}-\boldsymbol{q})$ in meson propagators are momentum-dependent as $||\boldsymbol{k}|-|\boldsymbol{q}||$ in numerical calculations.

Numerical convergences are determined at a given density and momentum when the difference of iterated values of single particle energies comes to $|E(\boldsymbol{k})-E'(\boldsymbol{k})|\leq 10^{-8}$. The numerical solutions can be obtained up to a high density, $k_F\sim 2.0$ fm^{-1} ($\rho_B\sim 0.540$ fm^{-3}), with reasonable iterations. In the applications to high density matter such as maximum masses of neutron stars, it is typically known that main contributions come from energy density within $k_F\sim 2.0$ fm^{-1} [1–9].

The CDHF binding energy, $\mathcal{E}_{HF}/\rho_B-M$, is shown in Fig. 2 and compared to the linear σ,ω Hartree (LHA) and nonlinear σ,ω Hartree (NHA) approximations [9]. The smooth curve of CDHF binding energy produces incompressibility $K=218$ MeV and symmetric energy $a_4=21.3$ MeV, at saturation density of nuclear matter. The HF approximation generates physically reasonable values compared to linear and nonlinear mean-field (Hartree) approximations. The properties of nuclear matter at saturation are listed in Table 1.

The energy densities of the direct (Hartree) and exchange (Fock) contributions are respectively shown in Fig 3. The dotted line is the Hartree portion of CDHF given by $\mathcal{E}_B+\mathcal{E}_H(\phi,\omega)$. The binding energy and saturation density are $\mathcal{E}/\rho_B-M=-26.58$ MeV, at $k_F=1.43$ fm^{-1} ($\rho_B=0.196$ fm^{-3}), indicated with an arrow. The direct interaction contributions saturate at a higher density than the normal density, $k_F=1.17$ fm^{-1} ($\rho_B=0.108$ fm^{-3}). The solid line is the exchange portion: $\mathcal{E}_B+\mathcal{E}_{EX}$. The binding energy and saturation density are $\mathcal{E}/\rho_B-M=-25.78$ MeV, at $k_F=1.17$ fm^{-1} ($\rho_B=0.108$ fm^{-3}), lower than normal saturation density.

The HF analysis suggests that the self-consistent HF approximation is a physically reasonable approximation to analyze ground state properties of nuclear matter than the mean-field

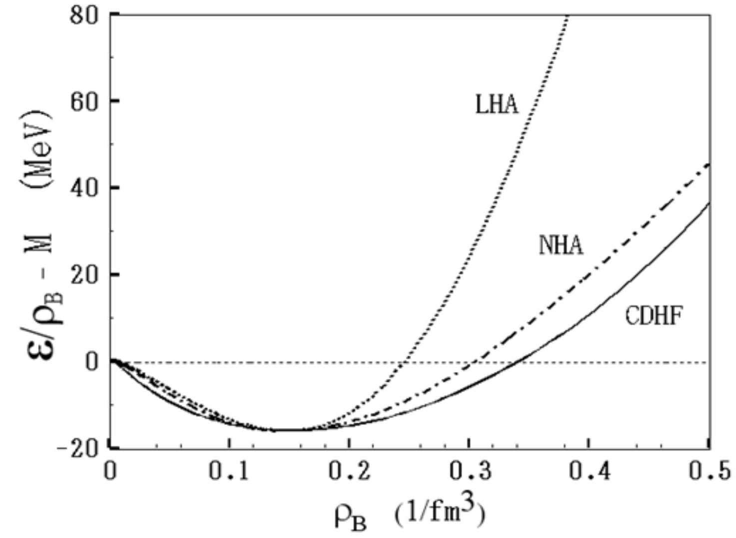

Fig. 2 Binding energies of symmetric nuclear matter at $T = 0$. The solid line is the chiral Dirac-Harteree-Fock (CDHF) approximation, which gives incompressibility $K = 218$ MeV and symmetry energy $a_4 = 21.3$ MeV. The dotted-line is LHA (linear σ, ω Hartree approximation, $K = 541$ MeV and $a_4 = 19.3$ MeV). The dot-dashed is NHA (nonlinear σ, ω Hartree approximation, $K = 333$ MeV and $a_4 = 15.3$ MeV) [9]. The saturation condition is: $\rho_B = 0.148$ (1/fm^3), $\mathcal{E}/\rho_B - M = -15.75$ MeV.

approximation, because Fock-exchange interactions appear first at low densities as quantum effects. The direct and exchange contributions altogether lead to binding energy of nuclear matter at saturation density, $k_F = 1.30$ fm^{-1} ($\rho_B = 0.148$ fm^{-3}), $\mathcal{E}/\rho_B - M = -15.75$ MeV (compare CDHF in Fig 2 and Fig 3). The mean-field interactions contribute prominently at high densities where classical pictures become dominant [1]. Whatever sophisticated nonlinear

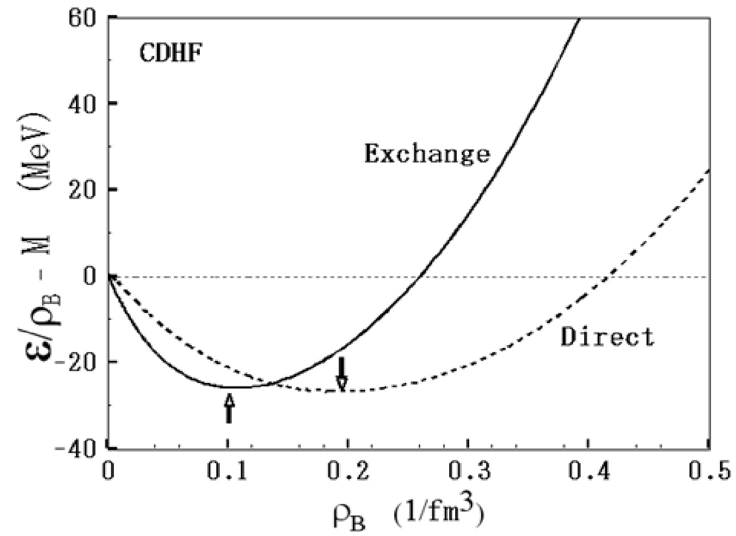

Fig. 3 The direct and exchange contributions to binding energy. Note the different saturation densities. The exchange interaction is more important at saturation than that of the direct (Hartree) interaction, whereas the direct interaction is dominant at high densities.

mean-field Lagrangian one can employ, the HF state is an appropriate ground state than that of the mean-field approximation so as to examine properties of nuclear matter at and above saturation density. The mean-field approximations should be extended to HF approximations to examine nuclear matter and high density matter.

The effective masses of nucleon at Fermi surface, $M^*_{HF}(k_F)$, are shown in Fig. 4. The CDHF calculation yields $M^*_{HF}(k_F)/M = 0.76$ at saturation density $k_F = 1.30\ \mathrm{fm}^{-1}$, whereas the linear (σ, ω) mean-field approximation (LHA) leads to $M^*(k_F)/M = 0.54$. The effective mass and other physical properties of nuclear matter are closely related to each other [7–9].

The incompressibility, K, is defined by:

$$K = 9\rho_B \frac{\partial^2 \mathcal{E}}{\partial \rho_B^2} \ . \tag{4.1}$$

The symmetry energy, a_4, is:

$$a_4 = \frac{1}{2}\rho_B \Big[\Big(\frac{\partial^2 \mathcal{E}}{\partial \rho_3^2}\Big)_{\rho_B}\Big]_{\rho_3=0} = \frac{k_F^2}{6E^*(k_F)} + \frac{1}{2}\rho_B \Big[\Big(\frac{\partial^2 \mathcal{E}_F}{\partial \rho_3^2}\Big)_{\rho_B}\Big]_{\rho_3=0} \ , \tag{4.2}$$

where the isovector density, ρ_3, is given by $\rho_3 = \rho_p - \rho_n = (k^3_{F_p} - k^3_{F_n})/3\pi^2$ using proton and neutron densities [13–15, 51], and the baryon density is $\rho_B = \rho_p + \rho_n = 2k_F^3/3\pi^2$ for symmetric nuclear matter. The symmetry energy is generally divided into direct and exchange contributions in thermodynamically consistent approximations, which is used in numerical evaluations. The CDHF approximation yields incompressibility, $K = 218$ MeV and symmetry energy, $a_4 = 21.3$ MeV.

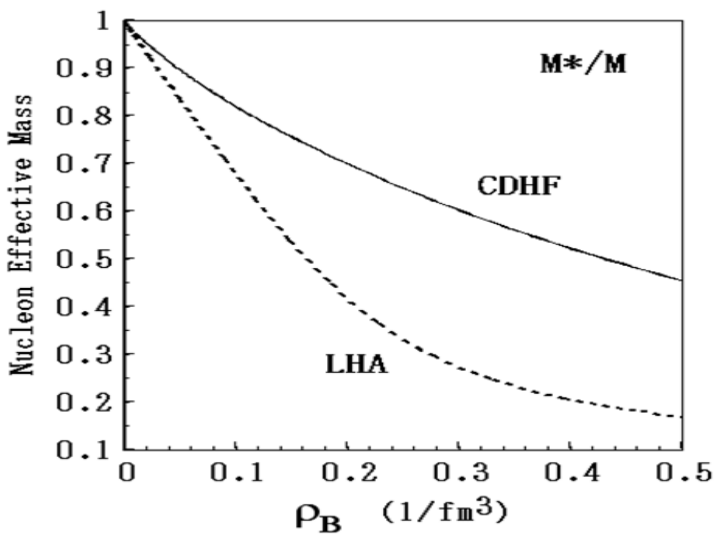

Fig. 4 The effective mass of nucleon. The solid line is the chiral Dirac-Harteree-Fock (CDHF) approximation. The effective mass of nucleon on the Fermi surface is, $M^*(k_F)/M = 0.76$, at saturation density, $k_F = 1.3\ \mathrm{fm}^{-1}$ ($\rho_B = 0.148\ \mathrm{fm}^{-3}$). The dotted-line is LHA (linear σ, ω Hartree approximation.

5 Properties of neutron stars

A neutron star is a high density object that is anticipated from the gravitational collapse of a massive star during a supernova explosion. Neutron stars are considered to be supported against further collapse with Pauli exclusion principle exerted by nuclear particles. The balance between gravitational force and quantum mechanical force by nucleons should be attributed to the existence of a high density neutron matter [13, 52–56]. Hence, properties of neutron stars such as mass, radius, central energy, and density region of stability, are expected to include information of nuclear dynamics and interactions.

The Einstein equation in the presence of matter is given by:

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -8\pi T^{\mu\nu} \ , \tag{5.1}$$

where $R^{\mu\nu}$ is Ricci tensor and $T^{\mu\nu}$ is symmetric two-index tensor indicating the presence of matter supposed to be the energy-momentum tensor defined by models of nuclear physics. The Tolman-Oppenheimer-Volkoff (TOV) equation is derived from the Einstein's equation (5.1):

$$\frac{dp}{dr} = -\frac{G\left[p(r) + \mathcal{E}(r)\right]\left[M(r) + 4\pi r^3 p(r)\right]}{r\left[r - 2GM(r)\right]} \ , \tag{5.2}$$

where G is inserted for a mnemonic purpose, and the mass of a sphere with the radius of r is defined by:

$$M(r) \equiv 4\pi \int_0^r dx x^2 \mathcal{E}(x) \ . \tag{5.3}$$

In a solid (or liquid) phase of nuclear matter, the edge of the star is sharply defined by disappearance of pressure at the surface of a star, $p(R) = 0$. Now, it is possible to calculate the maximum mass and radius of neutron stars with baryon density, pressure and energy density (equation of state, EOS) discussed in the previous sections. Constraints at nuclear matter saturation and thermodynamic consistency are essential to extract coherent results and examine correlations for low and high density nuclear matter. One should prove and display thermodynamic consistency before one should apply an approximation to nuclear and high density matter.

Table 1. The CDHF Fermi-liquid properties of nuclear matter at saturation and neutron stars with those of LHA (linear σ, ω mean-field) and CHA (chiral σ, ω mean-field) are listed. The parameters in the current CDHF approximation are fixed as, $g = 1.110$, $g_\omega = 8.712$, $m_\sigma = 70.0$ MeV, $m_\omega = 783.0$ MeV, $m_\pi = 138.0$ MeV.

	M_N^*/M	m_σ^*/m_σ	m_ω^*/m_ω	m_π^*/m_π	K (MeV)	a_4 (MeV)	$M_{star}^{max}/M_\odot$	R_{star} (km)
LHA	0.54	1	1	–	530	19.3	3.03	13.5
CHA	0.60	1.09	1.04	–	371	17.4	2.60	12.8
CDHF	0.76	1.32	1.05	1.02	218	21.3	2.21	11.6

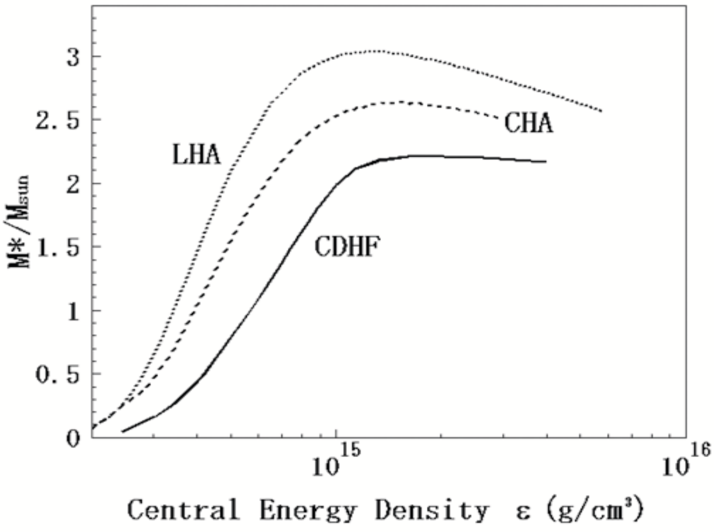

Fig. 5 The neutron star mass vs. central energy density. The solid line is the result of chiral Dirac-Harteree-Fock (CDHF) approximation, resulting in the mass of neutron stars: $M_{star}/M_{\odot} = 2.21$ (see, table 1). The dotted-line and dashed-line are respectively from LHA (linear σ, ω Hartree approximation) and CHA (chiral Hartree approximation).

The neutron star masses against central energy density are shown in Fig. 5. Masses and radii of neutron stars and properties of nuclear matter at saturation derived from the CDHF are listed in Table 1 and compared with those of linear (σ, ω) mean-field (LHA) [1] and chiral mean-field (CHA) [7, 8, 13] approximations. The CDHF results improve those of mean-field calculations, and extensions of self-consistent CDHF to more sophisticated approximations could contribute to better understanding of nuclear interactions.

The recently measured neutron star masses are reported as, $M^{max}_{star}/M_{\odot} = 1.97 \pm 0.04$, or $M^{max}_{star}/M_{\odot} = 2.01 \pm 0.04$ [57–60]. The empirical data are useful to scrutinize validity of nuclear models and approximations, however, the current self-consistent analysis and the new empirical data suggest that mean-field approximations to nuclear models are not sufficient to examine properties of nuclear matter at saturation and neutron stars, because exchange, vertex and retardation interactions may cause significant effects on EOS of nuclear matter. The current effective nuclear model and approximations should be extended to include complex many-body mechanism and retardation effects in order to extract physically consistent properties of hadron interactions.

6 Concluding Remarks

There are several important conclusions to declare in the current paper. Firstly, we emphasize that physically appropriate ground state of nuclear matter should be the Hartree-Fock ground state. As demonstrated in sec. 4, important contributions come from exchange interactions at saturation density. Many researches have investigated nuclear matter with complicated nonlinear mean-field approximations equivalent to Hartree approximations. The proof can be attained when nonlinear interactions are correctly renormalized as effective masses of mesons and baryons, effective coupling constants.

The significant difference between Hartree and Fock interactions comes from energy transfers in Fock exchange terms, which appears as retardation effects on physical quantities. In the relativistic approach, retardation effects will first emerge from the Fock exchange term, and several attempts have been made to obtain a self-consistent set of equations with retardation interactions. We have explicitly shown in the paper to derive a self-consistent set of equations for self-energies from the requirement of thermodynamic consistency. Retardation effects have generated a soft EOS, and HF calculations resulted in improved physical quantities at saturation and high densities.

Secondly, a set of functional integro-differential equations for self-energies are derived from conditions of self-consistency, and accurate solutions are compared with those constructed from Feynman diagram method. However, self-consistent solutions are different from those obtained by Feynman diagram construction. Specifically, retardation terms induced from energy-momentum tensor and nonlinear interactions are noticeably different [21–24]. This is an open question whether the difference comes from density-dependent Green's function approach defined by $G_D(k)$, or truncation of retardation interactions, *etc.* The investigation would be useful for many-body theory.

Finally, we discussed the relation between the condition of thermodynamic consistency and the density functional theory (DFT) in detail. Self-consistency would be considered as a method to integrate higher order corrections into one particle Green's function by way of renormalization, thus making higher order corrections small. The accuracy of self-consistency can be checked by the condition: $P_{hy} = P_T = P_{HV}$, and in case of nuclear physics, the additional condition, $P_{hy} = P_T = P_{HV} = 0$, is required at nuclear matter saturation. It is different from the artificial decoupling of equations of motion in hierarchical, infinite BBGKY chain problems [49, 50].

It is essential to include physically consistent vertex corrections into the current model, because the vertex modifications could generate additional retardation effects on entire calculations. Extensions to Bruckner HF, Ring relativistic approximations with appropriate vertex corrections which maintain the requirement of thermodynamic consistency should be investigated. The problems of self-consistency discussed in the paper would arise model-independently, and it is not confined to specific models in nuclear matter approximations. The theoretical analysis discussed in the paper would help understand the properties of ground-state of nuclear and high density matter.

References

[1] B. D. Serot and J. D. Walecka, *Advances in Nuclear Physics*, edited by J. W. Negele and E. Vogt (Plenum, New York, 1986), Vol. 16.

[2] Walecka, J.D. A Theory of Highly Condensed Matter. Annals of Physics, 83, (1974) 491.

[3] B. D. Serot, Quantum hadrodynamics, Reports on Progress in Physics, **55** (1992) 1855.

[4] Serot, B.D. and Uechi, H., Neutron Stars in Relativistic Hadron-Quark Models. Annals of Physics, 179, (1987) 272.

[5] B. D. Serot and J. D. Walecka, Recent Progress in Many-Body Theories, vol. 3, T. L. Ainsworth, C. E. Campbell, B. E. Clements, and E. Krotscheck, eds. (Plenum, New York, 1992), p. 49.

[6] B. D. Serot, Covariant effective field theory for nuclear structure and nuclear currents, Lcct. Notes Phys. **641** (2004) 31.

[7] S. T. Uechi, H. Uechi, Hardon-Quark Hybrid Stars Constructed by the Nonlinear σ, ω, ρ Mean-Field Model and MIT-Bag Model, Open Access Library Journal, 2, (2015).

[8] S. T. Uechi and H. Uechi, Density-Dependent Properties of Hadronic Matter in an Extended Chiral (σ, π, ω) Mean-Field Model, Open Access Library Journal, 2, (2015).

[9] S.T. Uechi and H. Uechi, Landau Theory of Fermi Liquid in a Relativistic Nonlinear (σ, ω) Model at Finite Temperature. Open Access Library Journal, 3: e2757 (2016).

[10] J. D. Walecka, *Theoretical Nuclear and Subnuclear Physics*, Oxford University Press, 1995.

[11] Serot, B.D. and Walecka, J.D., Chiral QHD with Vector Mesons. Acta Physica Polonica B, 23, (1992) 655.

[12] H. Müller and B. D. Serot, Relativistic mean-field theory and the high-density nuclear equation of stateNucl. Phys. **A606** (1996) 508.

[13] H. Uechi, S. T. Uechi, and B. S. Serot, B.D., *Neutron Stars: The Aspect of High Density Matter, Equations of State and Observables*. Nova Science Publishers, New York (2012).

[14] H. Uechi, Properties of Nuclear and Neutron Matter in a Nonlinear σ-ω-ρ Mean-Field Approximation with Self- and Mixed-Interactions. Nuclear Physics A, 780, (2006) 247.

[15] H. Uechi, Density-Dependent Correlations between Properties of Nuclear Matter and Neutron Stars in a Nonlinear σ, π, ω Mean-Field Approximation. Nuclear Physics A, 799, (2008) 181.

[16] L. D. Landau, Theory of Fermi-liquids, Zh. Eksp. Teor. Fiz. **30**, (1956) 1058, [Sov. Phys. JETP **3**, 920 (1957)].

[17] L. D. Landau, Oscillations in a Fermi-liquid, Zh. Eksp. Teor. Fiz. **32**, (1957) 59 [Sov. Phys. JETP **5**, 101].

[18] W. Kohn and L. J. Sham, Self-Consistent Equations Including Exchange and Correlation Effects. Physical Review, 140, (1965) A1133-A1138.

[19] W. Kohn, Nobel Lecture: Electronic Structure of Matter-Wave Functions and Density Functional. Reviews of Modern Physics, 71, (1999) 1253.

[20] C. J. Horowitz and B. D. Serot, Properties of nuclear and neutron matter in a relativistic Hartree-Fock theory, Nuclear Physics, **A399** (1983) 529.

[21] H. Uechi, Fermi-liquid properties of nuclear matter in a Dirac-Hartree-fock approximation, Nuclear Physics, **A501** (1989) 813.

[22] H. Uechi, Self-consistent structure in a relativistic Dirac-Hartree-Fock approximation, Nuclear Physics, **A696** (2001) 511.

[23] H. Uechi, The Effective Chiral (σ, π, ω) Model of Quantum Hadrodynamics Applied to Nuclear Matter and Neutron Stars, Journal of Applied Mathematics and Physics, 3, (2015) 114. Published Online February 2015 in SciRes.

[24] H. Uechi, The Theory of conserving approximations and the density functional theory in approximations for nuclear matter, Progress of Theoretical Physics, **111** (2004) 525.

[25] L. D. Miller, Exchange potentials in relativistic Hartree-Fock theory of closed-shell nuclei, Phys. Rev. **C 9**, (1974) 537.

[26] L. D. Miller, Relativistic single-particle potentials for nuclei, Annals of Physics, **V 91**, Issue 1, (1975) 40.

[27] S. Weinberg, Nonlinear Realizations of Chiral Symmetry, Phys. Rev. **166** (1968) 1568.

[28] S. Weinberg, Phenomenological Lagrangians, Physica, **A96**, (1979) 327.

[29] J. Schwinger, Chiral dynamics, Phys. Lett. **24B** (1967) 473.

[30] R. D. Puff, Groud-state properties of nuclear matter, Annals of Physics, **13**, (1961) 317.

[31] L. Wilets, R. D. Puff, D. Chiang and W. T. Nutt, Meson dynamics and the nuclear many-body problem. II. Finite density Hartree-Fock, Physical Review C **14**, (1976) 2269.

[32] A. F. Bielajew, and B. D. Serot, Renormalized Hartree-Fock equations in a nuclear relativistic quantum field theory, Annals of Physics, **V156**, Issue 2, Elsevier, (1984) 215.

[33] N. M. Hugenholtz and L. Van Hove, A theorem on the single particle energy in a Fermi gas with interaction, Physica **24** (1958) 363.

[34] B. D. Day, Current state of nuclear matter calculations, Rev. Mod. Phys. 50, (1978) 495.

[35] G. Baym and L. P. Kadanoff, Conservation Laws and Correlation Functions, Phys. Rev. **124** (1961) 287.

[36] G. Baym, Self-Consistent Approximations in Many-Body Systems, Phys. Rev. **127** (1962) 1391.

[37] M. Bonitz, R. Nareyka and D. Semkat, ed. *Progress in Nonequilibrium Green's Functions*, Proceedings of the conference, World Scientific, 2000.

[38] M. Bonitz, R. Nareyka and D. Semkat, ed. *Progress in Nonequilibrium Green's Functions II*, Proceedings of the conference, World Scientific, 2003.

[39] Y. Takada, Exact self-energy of the many-body problem from conserving approximations, Physical Review **B52**, (1995) 12708.

[40] G. Stefanucci and R. Van Leeuwen, Nonequilibrium Many-Body Theory of Quantum Systems, Cambridge University Press, (2013).

[41] N. P. Landsman and Ch. G. Van Weert, Real- and imaginary-time field theory at finite temperature and density, Phys. Reports **145** (1987) 141.

[42] H. Uechi, Constraints on the self-consistent relativistic Fermi-sea particle formalism in the quantum hadrodynamical model, Physical Review, **C41** (1990) 744.

[43] K. A. Brueckner and C. A. Levinson, Interacting Particles to a Problem of Self-Consistent Fields, Phys. Rev. **97** (1955) 1344

[44] K. A. Brueckner, Single Particle Energies in the Theory of Nuclear Matter, Phys. Rev. **110** (1958) 597.

[45] K. A. Brueckner and D. T. Goldman, Single Particle Energies in the Theory of Nuclear Matter, Physical Review, **117** (1960) 207.

[46] K. A. Brueckner, J. L. Gammel and J. T. Kubis, Calculation of Single-Particle Energies in the Theory of Nuclear Matter, Physical Review, **118** (1960) 1438.

[47] G. Baym and C. Pethick, in The physics of liquid and solid helium, Part II, ed. K. H. Bennemann and J. B. Ketterson (John Wiley, New York,1978).

[48] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems*, Dover Pub. New York (2003).

[49] M. Bonitz, *Quantum Kinetic Theory*, B.G. Tehbner Stuttgart, Leipzig, (1998).

[50] A. M. Zagoskin, *Quantum Theory of Many-Body Systems*, Springer, New York, Inc. (1998).

[51] B. D. Serot, A relativistic nuclear field theory with π and ρ mesons, Phys. Lett. **86 B** (1979) 146 [Erratum: Phys. Lett. **87B** (1979) 403].

[52] H. Heiselberg and M. Hjorth-Jensen, Phases of dense matter in neutron stars, Phys. Rep., **V328**, (2000) 237.

[53] J. M. Lattimer and M. Prakash, Neutron star observations: Prognosis for equation of state constraints, Phys. Rep., **442**, (2007) 109.

[54] R. P. Drake, *High-Energy-Density Physics*, Springer-verlag, (2006).

[55] S. Weinberg, *Gravitation and Cosmology*, John Wiley and Sons, New York, 1972.

[56] C. W. Misner, K. S. Thorne and J. W. Wheeler, *Gravitation*, W. H. Freeman and Company, New York, 1973.

[57] C. Constantinou, et.al., Thermal properties of hot and dense matter with finite range interactions, Physical Review, **C 92**, 025801 (2015).

[58] J. M. Lattimer and A. W. Steiner, Neutron Star Masses and Radii from Quiescent Low-Mass X-ray Binaries, The Astrophysical Journal, 784:123 (2014).

[59] P. B. Demorest, T. Pennucci, S. M. Ransom, M. S. E. Roberts, and J. W. T. Hessels, A two-solar-mass neutron star measured using Shapiro delay, Nature (London) 467 (2010) 1081.

[60] J. Antoniadis et al., A Massive Pulsar in a Compact Relativistic Binary, Science 340, (2013) 6131.

