RESEARCH ARTICLE

Coordinate Reference Systems and Gravity

Alessandro Caporali

Author's affiliation:

Università di Padova – Dipartimento di Geoscienze – Via Gradenigo 6 – 35137 Padova

* **Corresponding address:** E-mail: alessandro.caporali@unipd.it; Tel. 049 8279122

Abstract

Reference systems and gravity are topics relevant not only to Geodesy, but also to several other fields of research. The Theory of General Relativity is based on a geometric view of gravity, and therefore coordinates and gravity become inseparable. As the precision of the geodetic measurements with satellite techniques increases, the classical measurement modeling progressively gives way to a treatment in which concepts of Relativity take on an ever greater importance. In this paper, the differences between the classical and relativistic concepts of space-time in the presence of gravity, and the corresponding different role of the coordinates are first analyzed. Then, we move on to the application plan, facing the three major - for the moment - examples of the presence of General Relativity in the representation of geodetic measurements: secular and periodic modification of the time delivered by on-board atomic clocks in GNSS satellites; deviation of the trajectory of light rays in a gravitational field, and corresponding delay in arrival time; and finally dragging of the inertial systems due to the rotation of the central body (Lense Thirring effect).

1. Introduction

The aim of this paper is to review the most important aspects of the relationship between gravity and coordinates, in consideration of the ever growing presence of the concepts of General Relativity in Geodesy. The paper is organized into a first review of the Classic and Relativistic concepts of coordinates and fields. An analysis of the observable effects of gravity on reference frames and coordinates is made in the following section. To properly model the observables of Satellite Geodesy, the classical approach needs to embody the curvature of Space Time in the temporal and spatial dimensions. The rotation of the central body further introduces a coupling of the spatial and temporal dimensions, which is also in principle observable. The conclusion is that the classical approach to coordinate reference frames must progressively be replaced by a vision of coordinate reference frames where gravity affects both the spatial and the temporal dimension. Concepts on General Relativity and Gravitation are

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discussed in detail in classical text books such as Misner et al. (1975). A comprehensive review of relativistic effects in satellite data is given by Ashby (2003) and Friedman (2017), for example.

2. Coordinates and Field Theory

In the Euclidean Space Time, Space and Time are orthogonal: time is an independent scalar whereas spatial coordinates are measured along three mutually orthogonal axes. The squared distance between two points is, in Cartesian coordinates,

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{1}$$

A test particle not subject to a force travels along a straight line with constant velocity. Hence, the trajectory retains its shape in all reference frames, which move at a constant speed relative to each other.

In Special Relativity, time becomes a coordinate, and the four dimensional distance between two event points can be positive, negative or zero depending on whether the two events are outside, inside or on the null cone. Lorentz transformations replace the transformation between two inertial systems. Both the spatial and temporal coordinates transform in such a way that the four dimensional distance between two neighboring event points is invariant:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{2}$$

where

$$\eta_{\mu\nu} = \mathrm{diag}\,(-1,1,1,1) \tag{3}$$

is the Minkowski metric tensor of Special Relativity. Repeated indexes imply summation.

In the Riemannian geometry of General Relativity, the squared distance between two neighboring event points is:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{4}$$

The infinitesimal increments dx^μ (μ=0..3) of coordinates together with the metric tensor $g_{\mu\nu}$ define a quadratic form which, like in Special Relativity, can be positive, negative or null depending on whether two events separated by the spacetime squared distance ds^2 are respectively inside, outside or on the null cone which separates causally from non causally related events.

In Einstein's General Relativity, the metric tensor is a function of the coordinates. Hence two neighboring event points separated in space time by small increments in their coordinates have a four dimensional distance *ds* which can be positive, negative or zero, depending on the local value of the metric tensor, and ultimately on the curvature of space time, or gravity. This situation is not uncommon: for example in cartography the linear deformation modulus does depend on the position, so that if two pairs of points have the same distance on the map projection, they may have different distance on the projected surface (e.g. the surface of Earth), depending on their position.

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In Classical Theory, the fields act in a flat three-dimensional space. The equations of the field and the equations of motion are distinct from each other, and each requires its own boundary conditions. If U(x,y,z) is a gravitational potential, the field equations and the equation of motion are respectively:

$$\nabla^2 U = 4\pi G\rho; \quad m\frac{d^2\vec{r}}{dt^2} = -m\nabla U \tag{5}$$

where m is the mass of a test particle, G is the universal constant of gravity and ρ is the mass density of the body creating gravity. The vector **r** points from the origin of the coordinate system to the test mass. The coordinates are rectilinear. The coordinate time t is an independent variable along the path and remains unchanged under coordinate transformations, i.e. it is a scalar. The field equations do not imply the equations of motion.

In General Relativity, Space Time has a curvature defined in terms of the metric tensor and of its derivatives. The covariant derivative is such that a) the covariant derivative of the metric tensor is zero, and b) the equation of a body freely moving under the action of gravity is that of a geodesic:

$$g^{\mu\nu}{}_{;\nu} \equiv \frac{\partial g^{\mu\nu}}{\partial x^\nu} + \Gamma^{\mu}_{\alpha\beta} g^{\alpha\beta} + \Gamma^{\nu}_{\alpha\beta} g^{\alpha\beta} = 0; \; \frac{D^2 x^\mu}{ds^2} \equiv \frac{d^2 x^\mu}{ds^2} + \Gamma^{\mu}_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \tag{6}$$

$$\text{where } \Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda}\left(\frac{\partial g_{\alpha\lambda}}{\partial x^\beta} + \frac{\partial g_{\beta\lambda}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\lambda}\right) \tag{7}$$

The semicolon (;) indicates the covariant derivative. While the metric tensor is a function of the coordinates, its covariant derivative is zero, in the same way as the Minkowski metric tensor of Special Relativity is independent of the coordinates. Likewise, the idea that a test particle moving under the action of gravity follows a geodesic is the analog of the fact that in Special Relativity or in classical mechanics an undisturbed test particle moves along a straight line.

Einstein's equations do imply the equations of the motion of matter (the relativistic generalization of the Navier Stokes equation):

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^2} T_{\mu\nu} \dashrightarrow T^{\mu\nu}_{;\nu} = 0 \tag{8}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is its trace, $T_{\mu\nu}$ is the moment energy tensor. 'c' is the speed of light in vacuum.

Einstein equations form a set of hyperbolic differential equations in the components of the metric tensor. Exact, asymptotically Minkowskian solutions ($g_{\mu\nu} \rightarrow \eta_{\mu\nu}$ at large spatial distances) exist (e.g. Schwarzschild, Kerr, Tomimatsu & Sato). For applications in cases of practical interest (weak field, small velocities) a Post Newtonian approximation is sufficient (Caporali, 1981 a,b). Without going into the details of the metric tensor coefficients, it can be observed that in the Post Newtonian approximation the term g_{00} contains the gravitational potential and its square; the term g_{0i} (i = 1,3) depends on the vector gravitational potential and is the analog of the vector potential A of the electromagnetism; the term g_{ij} depends on the gravitational potential.

Not all Einstein equations are of the second order in the metric tensor, and therefore some of

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them are constraints on the boundary conditions, e.g. that Space-Time is asymptotically Minkowskian.

3. How gravity and Space Time coordinates interact

A reference system is basically a base with graduated rods and a sample of time, with a given origin and orientation in space and time. In general, the mathematical form of the laws of Nature, and thus the interpretation of measures, depends on the reference system. For example, the motion of satellites around the Earth can be described in a reference system aligned with the 'fixed stars', or inertial, or in a reference system with axes rotating together with the Earth, or ECEF (Earth Centered, Earth Fixed). Since the Earth is in accelerated motion around the center of gravity of the Solar System, inertiality is only locally valid, but this is compatible with the principles set out in the previous paragraph. The equations of motion of a satellite assume a different form in Classical Theory if formulated in an inertial system or in an ECEF system. In fact, the covariance is guaranteed only between inertial reference systems.

Similarly for the propagation of electromagnetic signals, D'Alambert's wave equation is covariant under Lorentz transformations, hence only in the context of the Special Relativity. However, the same equation becomes covariant for arbitrary coordinate transformations, when the covariant derivative is substituted for the ordinary derivative.

The practice of Satellite Geodesy, with its ever increasing precision, has placed the Euclidean Geometry at its limits, highlighting the need to introduce some corrective terms that are justified only in the Theory of General Relativity. In Satellite Geodesy with satellites in orbit at distances equal to several terrestrial radii, the hypothesis of locality is invalid. In the following, we will analyze some particularly interesting aspects related to the modeling of GNSS (Global Navigation Satellite System) data.

3.1. Synchronism between a space borne atomic oscillator and a ground based oscillator

In the analysis of the GNSS data, the transmission time of a wave packet, measured on the scale of the on-board atomic oscillator, is compared with the reception time of the centroid of the wave packet, measured on the scale of the oscillator in the receiver. The two time coordinates are such that:

a) both the on-board transmitter and the ground receiver define their time in non-inertial systems, i.e. accelerated;

b) the on-board transmitter is in variable gravity conditions due to eccentricity of the orbit, whereas the receiver on the Earth surface is at a constant gravity.

Point a) implies that the coordinates of the transmitter and of the receiver are defined in reference systems which are rotated relative to each other by an angle equal to the flight time times the terrestrial angular velocity ω_e. Point b) implies that the proper time maintained by an

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atomic oscillator on board the satellite changes its frequency as the orbital phase changes and therefore the local gravitational potential, i.e. the curvature of the local space-time.

The following model of the pseudo distance p (t), in the sense of difference between time of arrival t on the receiver clock and time of departure t ' on the onboard clock of the wave packet, realizes how the two points listed above can be taken into account:

$$p(t) = cdt(t') + \sqrt{[X(t') + \omega_e Y(t-t') - x]^2 + [Y(t') - \omega_e X(t-t') - y]^2 + [Z(t') - z]^2} + cdT(t) \quad (9)$$

The term dt(t’) is the misalignment of the transmitter clock from a common time scale. It consists of a nearly linear drift plus a nearly sinusoidal term which describes the variation along the orbit of the gravity, due to the orbital eccentricity:

$$dt(t') = a_0 + a_1(t' - T_{oc)} - 2\frac{\vec{r}\cdot\vec{v}}{c^2} \quad (10)$$

Where Toc is a reference time for the clock polynomial and the last term is the General Relativistic modulation caused by the orbital eccentricity.

The second term shows that the geometrical travel time involves coordinates of the satellite (capital letters) and receiver defined in reference frames rotated relative to each other about the z-axis. Hence the correction term dependent on the product of the Earth angular velocity by the time of flight:

$$t' \approx t - \sqrt{(X-x)^2 + (Y-y)^2 + (Z-z)^2}/c \quad (11)$$

The last term dT(t) describes the offset of the receiver time relative to the common time scale.

It should be noted that the nominal frequency of the on-board oscillator must be decreased in percentage by the variation in energy (kinetic + potential), or $4.464 * 10^{-10}$, compared to that on the Earth's surface, such that once in orbit, and therefore in conditions of lower gravity, increases and returns to the nominal value.

$$\frac{\Delta f}{f} = \frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{\Delta U}{c^2} \cong -4.464\ \ 10^{-10} \quad (12)$$

We assumed an orbital radius of 26000 km and an orbital velocity of 3.9 km/sec. ΔU is the difference between the potential on the Earth surface and at the orbital height. If the on board clock is expected to work at a frequency of 10.23 MHz while in orbit, it needs to be calibrated on the Earth surface at a frequency of $4.464*10^{-10}$ * 10.23MHz= $4.57*10^{-3}$ Hz lower, that is 10.22999999543 MHz. Once in orbit, the lower gravity implies a correspondingly higher frequency or 10.23 MHz.

Likewise, the scale factor between a length unit at orbital altitude and at the Earth surface is of the same order of magnitude. This value is at the threshold of the sensitivity when comparing different spatial reference frames by means of a similitude or Helmert transformation involving a scale factor, three rotations and three translations.

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3.2. Light propagation along trajectories curved by gravity

Previously, it has been mentioned that in Special Relativity light propagates along a straight line. From the point of view of General Relativity, instead, light follows a geodetic trajectory of zero length in Space-Time. Therefore, if Space-Time is curved to account for gravity, also the geodesics followed by the light rays will follow a different trajectory from that of the Minkowskian Space-Time. The curvature of Space-Time therefore has the double effect of:

a) delaying the arrival time of a signal, because at equal speed the optical path has lengthened with respect to what would be followed in Minkowskian Space-Time

b) move the apparent position of the light source, as the gravitational field behaves like a lens.

The time delay in a) is the so called Shapiro effect (Shapiro, 1964), which for Earth orbiting satellites can be of the order of 0.02 m. It is computed using the expression:

$$\Delta\rho = \frac{2GM}{c^2} ln \frac{R+r+|R-r|}{R+r-|R-r|} \tag{13}$$

Where R, r are the distance from the center of gravity of the satellite and respectively the ground receiver.

Likewise, the deflection of light grazing a central body of radius r (Einstein, 1911) is:

$$\Delta\varphi = \frac{2GM}{c^2 r} \tag{14}$$

This deflection for a wave reaching the Earth implies a change in elevation exceedingly small, of the order of a fraction of milliarcsecond. In practice, radio-occultation experiments are widely used, not so much for purposes of testing General Relativity but for measuring the properties of the Earth's troposphere, which produces a curvature effect that dominates the Relativistic one.

3.3. Non-orthogonality of the temporal dimension with respect to the spatial dimension

In Special Relativity, the three-dimensional space is Euclidean, and the spatial and temporal dimensions are orthogonal, as shown by the cancellation of the mixed terms g_{0i}.

The latter term is different from zero in the first Post Newtonian approximation: it is proportional to a potential gravitational vector and accounts for a 'gravito-magnetic' property unknown in classical gravity, but which has obvious similarities with electromagnetic theory. As moving charges can produce a magnetic field, so moving masses can produce in general a completely similar 'gravito-magnetic' field. Experimental evidence is the Lense and Thirring precession of the line of the nodes of a satellite in orbit around a rotating Earth (Pfister, 2017).

If the metric tensor is non-diagonal in the space-time component, then the trajectory of a radially falling test particle will wind up following the rotation of the central body that generates the field. The law of motion is mathematically identical to that of a freely falling body in a rotating system:

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$$r\frac{d^2\varphi}{dt^2} + 2\omega\frac{dr}{dt} = 0 \quad (15)$$

where φ is the angular variable or longitude of the falling body. In classical mechanics, ω is the angular rotation of the frame, so that the Coriolis force vanishes in a non rotating frame. In General Relativity

$$\omega = \frac{GS}{c^2 r^3} \quad (16)$$

where S is the angular momentum or spin of the central body, e.g. the Earth. Hence the bending of a trajectory is visible in a non rotating frame.

The equation of motion of a free falling body in the gravity field of a spinning body can be represented in a classical way defining the scalar and vector gravitational potentials respectively U and **A:**

$$U = -\frac{GM}{r}\ ;\ \vec{A} = \frac{G\vec{S}\times\vec{r}}{r^3 c} \quad (17)$$

Then the fields are computed as usual in classical electromagnetism:

$$\vec{E} = -\nabla U\ ;\ \vec{B} = \frac{1}{2}\nabla\times\vec{A} \quad (18)$$

And the equation of motion of the test particle in the gravitomagnetic field of a central body of mass M and spin S has the form:

$$m\frac{d^2\vec{r}}{dt^2} = m\vec{E} + 4m\frac{d\vec{r}}{dt}\times\vec{B} \quad (19)$$

This effect is known as 'dragging of inertial frames'. The order of magnitude is $(v / c)^3 \cong 10^{-15}$, that is five orders of magnitude smaller than the light bending and the stretching of space and time discussed in the two previous subsections, and about three orders of magnitude below the scale factor estimated in the Helmert transformations used to monitor the stability of the operational realization of terrestrial reference systems.

4. Conclusions

Gravity and reference systems have traditionally been strictly bound in the case of dimension measurements that refer to an equipotential surface, such as height. Geodesits often express the gravity related heights in units of Geopotential numbers, which have dimension of energy and are defined as a difference of gravitational potential. The planimetric coordinates have instead always been considered as free from the effect of gravity. Not so in General Relativity, where gravity is expressed as a curvature of Space-Time. Therefore, the Space-Time coordinate lines embody these curvature properties. Geodesists are consequently witnessing a moment of transition from a Classical to a Relativistic way of considering coordinates and gravity, thanks also to the availability of experimental measures of great precision and sensitivity, and to the fact that Geodesy assumes an increasingly global value in both science and applications.

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