Inviscid 1 D fluids under gravity

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Abstract

One-dimensional, inviscid, compressible, isothermal and isentropic fluids under gravity are considered as usefull preambles relevant to the theory of uni-axial meteorological phenomena [$4, Sect.4, items$ 3 & 4]. In Sect.1, the continuity equation, the Euler equations of these fluids, the equations of their charcteristics as well as those of their energies, constants of the motion, are given. Next, the continuy and Euler equations for isentropic fluids are written in a matrix and compact forms, their diagonalized versions are shawn to be total derivatives of new constants of the motion, identified as gravitational Riemann invariants, [1] .Then, and in Sect.1, also, the mass densities at time t occuring in the above equations are expressed as product of their initial value times the inverse Jacobian of the characteriistics of the fluids with respect to their initial values, an operation permitting to generate, central in this work, the first order non-linear partial differential equations satisfied by these invariants and, also, those of the other constants of the motion, the energies. In Sect.2, the Charpit scheme, designed to solve non-linear first order PDE's of n variables, in general, [$2, ch.4$] , is presented. For systems of two independant variables, the correspding ordinary differential equations are given in Sect.2.1.The Charpit functions for the Riemannian cases and for the other cases are given in Sect.2.2.Then,in Sect.3, the attention is focussed on the gravitational Riemann invariants only, owing to their originality and also, to the relative simplicity of the numerics implied. Their corresponding ODE's are given in Subsect.3.1 and, in Subsect.3.2, several examples are propposed and some of their explcit solutions, algebraic and graphical are reported .To conclude, due comparaison is made between these results and solutions of equivalent PDE's, given in [3, $No.2.1.2.3, p.45$] .

Highlights...

°1D inviscid, cimpressible, isothermal or isentropic fluids under gravitation.

°Gravitational Riemann Invariants for isentropic systems

°Charpit scheme, EDO's and examples of isentropiic solutions

Keywords

1 Dimensional,inviscid fluds,gravitational Riemann Invariants

1 Introduction.

Assuming, for simplicity, inelastic absorption of the fluid particles on the ground, the coordinate of the characteristics is $z(t) \geq 0$ and also $< \infty$,w hereas the time variable t is choosen to be ≥ 0. The governing PDE's of these fluids are: the continuity equation, with $\rho(z,t) \geq 0$ for the mass density,and $u(z,t) \subset R^1$for the velocity field, i.e.

$$\partial\rho/\partial t + \partial\left(\rho u\right)/\partial z = 0 \tag{1}$$

and the Euler equation

$$\partial u/\partial t{+}u\partial u/\partial z{+}g{+}\frac{1}{\rho}\partial P(\rho)/\partial z = 0, \tag{2}$$

where g is the gravitational constant and $P(\rho)$ is the pressure.

In the isothermal case we have ($k_B = Boltzmannconst., T = temperature$, $m = mass\ of\ a\ fluid\ particle$) and with the suffix T standing for isothermal,

$$P := P_T = k_B T\rho/m := c_T(T)^2\rho, \tag{3}$$

$c_T(T)$ being the temperature- dependant sound velocity and $\frac{1}{\rho}\partial P_T(\rho)/\partial z = c_T(T)^2\ \partial \ln\rho/\partial z$.

In the isentropic case,with C_p and C_V being respectively the specific heat of perfect gases at constant pressure and volume, with the suffix S standing for isentropic, with $P := P_S = Const.\rho^\nu/\nu$,where $\nu = C_P/C_V = (d/2+1)/d/2 = 3$ for $d = 1$ and $Const = k^2$,with $k^2\rho^2 := c_S(\rho)^2, c_S(\rho)$ being the density-dependant sound velocity, we have

$$P_S = k^2\rho^3/3 = c_S^2(\rho)\rho/3 \tag{4}$$

and $\frac{1}{\rho}\partial P_S(\rho)/\partial z = k^2\rho\partial\rho/\partial z = k^2\rho^2\partial\ln\rho/\partial z = c_S(\rho)^2\partial\ln\rho/\partial z$.

It is convenient to rewrite (2) for the two cases. We have, respectively,

$$\partial u/\partial t{+}u\partial u/\partial z{+}g + c_T(T)^2(\partial\rho/\partial z)/\rho = 0. \tag{5}$$

and

$$\partial u/\partial t{+}u\partial u/\partial z{+}g + c_S(\rho)^2(\partial\rho/\partial z)/\rho = 0. \tag{6}$$

From (5 & 6) , with $u = dz/dt =: \dot{z}$, and $\partial u/\partial t{+}u\partial u/\partial z = d^2z/dt^2 =: \ddot{z}$, we obtain the equations of motion of the characteritics of our models,written in parallel, namely (7)

$$\ddot{z} + g + \begin{pmatrix} c_T(T)^2 \\ c_S(\rho)^2 \end{pmatrix}(\partial\rho/\partial z)/\rho = 0, \tag{7}$$

and their constants of the motion, i.e.their energies, which read (8), with y

being the initial value of z,

$$\frac{1}{2}\dot{z}^2 + gz + c_T(T)^2 \ln\rho(z) = \frac{1}{2}u(y)^2 + gy + c_T(T)^2 \ln\rho_0(y), \tag{8}$$

for the isothermal case and (9) for the isentopic one,

$$\frac{1}{2}\dot{z}^2 + gz + \frac{1}{2}c_S(\rho(z))^2 = \frac{1}{2}u(y)^2 + gy + \frac{1}{2}c_S(\rho_0(y))^2. \tag{9}$$

We proceed in showing that, in addition to the conservation of energies recalled above, and in the isentropic case only, there are two other constants of the motion, closely related to the Riemann Invariants established in 1858 [1] for perfect, invsicid, compressible, and isentropic fluids in 1D ($cf.$ [2], sec$ts.$ 6.8.& 6.9 $for\ an\ extensive\ presentation$). In order to generalize these Invariants for $g > 0$, it is suitable to introduce the colomn vector $V =:\binom{\rho}{u+gt}$,the matrix $\mathbf{A}_S =: \begin{pmatrix} u & \rho \\ c_S(\rho)^2/\rho & u \end{pmatrix}$,and to rewrite (1) & (6) in the matrix and also compact forms

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ u+gt \end{pmatrix} + \begin{pmatrix} u & \rho \\ c_S(\rho)^2/\rho & u \end{pmatrix}\frac{\partial}{dz}\begin{pmatrix} \rho \\ u+gt \end{pmatrix} := \frac{\partial}{\partial t}V + \mathbf{A}_S\frac{\partial}{\partial z}V = \mathbf{0} \tag{10}$$

The eigenvalues of the 2 *by* 2 matrix are, in setting $\epsilon = +/-1$, $\lambda_S = u + \epsilon c_S(\rho)$ and its eigenvectors are $\vartheta_{S,\epsilon} := \binom{1}{\epsilon c_S/\rho}$. The unique property of these eigenvectors is that they are constant vectors since $c_S/\rho =$ constant k. A similar operation made with the isothermal case would have produced density dependant eigenvectors $\vartheta_{T,\epsilon} := \binom{1}{\epsilon c_T/\rho}$. With the matrix $\mathbf{M}_{S:} = (\ \vartheta_{S,+1}\ \vartheta_{S,-1})$,we can introduce the diagonalized version

$$V = \mathbf{M}_S W\ . \tag{11}$$

It follows that,with $\mathbf{A}_S\mathbf{M}_S = \mathbf{M}_S\lambda_S$,and since $\mathbf{M}_S =$ constant matrix $\neq \mathbf{0}$, we have that, $\frac{\partial}{\partial t}V + \mathbf{A}_S\frac{\partial}{\partial z}V = \mathbf{M}_S(\partial W/\partial_t + \lambda_S \partial W/\partial z) = \mathbf{0}$, or, explicitly,

$$\partial W/\partial_t + \lambda_S \partial W/\partial z = eq(6) + (\epsilon c_S/\rho)eq(1) = \mathbf{0}. \tag{12}$$

In terms of indefinite integrals, the solutions are

$$W(z,t;\epsilon) = u + gt + \epsilon\int^{\rho(z)} d\rho' c_S(\rho')/\rho' = u + gt + \epsilon c_S(\rho) \tag{13}$$

In the sequel, the equations $W(z,t;\epsilon) = W(y,0;\epsilon)$ will be needed. In fact $W(y,0;\epsilon)$ are, strictly speaking, the invariants, as of now identified as gravitational Riemann invariants. In terms of Lagrangian variables, their algebraic forms are ,

$$W(y,0;\epsilon) = u(y) + \epsilon c_S(\rho_0(y) = \dot{z} + gt + \epsilon c_S(\rho(z,t) = W(z,t;\epsilon) \tag{14}$$

Notice, here, that the solutions of $W(z,t;\epsilon))$ are also valid if gt is replaced by any differentiable external velocity field $u_e(t)$,as in the case of friction-less electric wires subject to time-dependant electric fields.

At this point we solve formally the continuity equation which expresses the conservation of mass as $\rho(z,t)dz = \rho(z(y,t),t)dz = \rho_0(y)dy$ by the introduction of the Jacobian $\partial z(y,t)/\partial y$, in setting (15)

$$\rho(z,t) = \rho(z(y,t),t) = \rho_0(y)(\partial z(y,t)/\partial y)^{-1}, \tag{15}$$

$\rho_0(y)$ being the, spatially dependant, initial density of the models. Notice that the positivity of $\rho(z,t)$ implies the positivity of the Jacobian $\partial z(y,t)/\partial y$. Using the fact that

$$\dot{z}(y,t) := \partial z(y,t)/\partial t, \tag{16}$$

and, with the simplifying notation $\partial z(y,t)/\partial y :=z_y$ as well as $\partial z(y,t)/\partial/t := z_t$,we can write (14),in function of the original variables and in terms of the followong first order non-linear PDE's, central in this work, namely (17)

$$z_t + \epsilon c_S(y)(z_y^{-1} - 1) + gt - u(y) = 0. \tag{17}$$

Clearly,in both cases, gt could be replaced by $u_e(t)$.

Notice that, for completness and with the same conventions of notation, the constants of the motion (8 & 9) can also be written as non-linear first order PDE's, quadratic in z_t, namely (18)

$$\frac{1}{2}z_t^2 + c_T(T)^2 \ln z_y - \frac{1}{2}u(y)^2 + g(z-y) = o, \tag{18}$$

and (19)

$$\frac{1}{2}z_t^2 + \frac{1}{2}c_S\ (y)^2 \left(z_y^{-2} - 1\right) - \frac{1}{2}u(y)^2 + g(z-y) = o, \tag{19}$$

We are ready to introduce the Charpit scheme for solving the nonlinear first order PDE's:$(17 : \textit{the Riemannian type})$ and,in principle also :$(18,19)$.However, as pointed out in the Introduction,the analysis and solutions of the *Riemannian* cases, only, are going to be dealt with in details here,whereas the general frameswork of discussion will be given for both types.The labeling of the variables will be (q_0,q,p_0,p,z) corresponding to $(c_0t := s, y, z_{c_0t}, z_y, z)$, where c_0 will be the temperature- dependant $c_T(T)$ or intial coordinate dependant (via the initial density) $c_S(y) := c_S(\rho_0(y))$ reference velocities, as specified above. The mapping of the above equations to those of the Charpit scheme , presented below, will require that y which is here, a prescribed, initial condition, becomes a variable, denoted $q \subset R^+$,of space-like nature and the solution of the corresponding first order Charpit ODE's will imply the provisional introduction of an initial value for q, denoted $r \subset R^+$. Ultimately, r will be eliminated as a function of q, i.e. of y and of the parameter s of the characteristics, a procedure checked in comparing our solutions of the *Rimannian* case, given in Subsect.3.2 with those of equivalent PDE's given in $[3, No.2.1.2.3, p.45]$

2 The Charpit scheme

The purpose of this scheme is to convert first order, nonlinear PDE 's of, say, 2n independant variables to a set of 2n ODE's $[2, ch.4]$,a purpose similar to but more general than that of Hamilton-Jacobi's scheme in analytical mechanics.The case of two independant variables is treated in details in the following two subsections.

2.1 The Charpit Equations

For a system of two independant variables and following a standard notation $[2, ch.4]$, let $F(x, y, p, q, u(x, y)) = 0$ be the said PDE ,or Charpit function, with $p = u_x$ $q = u_y$ and with the labeling $(x, y, p, q, u(x, y))$ corresponding to $(q_0, q, p_0, p, z(q_0, q))$,as used above.Let s be the parameter of the trajectories. The Charpit equations then read:

$$dx/ds = \partial F/\partial p := F_p \tag{20}$$

$$dy/ds = F_q \tag{21}$$

$$du/ds = pF_p + qF_q \tag{22}$$

Two more equations, for p and q, are needed. In fact, one only, for q, is needed, since that for p can be obtained from the Charpit function. Consider first

$$dF/dx = F_x + F_p p_x + F_q q_x + F_u u_x = 0 \tag{23}$$

and

$$dF/dy = F_y + F_p p_y + F_q q_y + F_u u_y = 0\ , \tag{24}$$

and take into account the integrability hypothesis

$$p_y = u_{x,y} = u_{y,x} = q_x\ . \tag{25}$$

Then, since $p_y = q_x$,we have

$$dF/dy = F_y + F_p q_x + F_q q_y + F_u u_y = 0\ , \tag{26}$$

and, symmetricaly, with $q_x = p_y$:

$$dF/dx = F_x + F_p p_x + F_q p_y + F_u u_x = 0\ . \tag{27}$$

It follows that

$$dq/ds = q_x F_p, + q_y Fq = -(F_y + F_u u_y)\ , \tag{28}$$

and that

$$dp/ds = p_x F_p + p_y F_q = -(F_x + F_u u_x). \tag{29}$$

We have also, in a compact form:

$$ds = \frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{u_x F_p + u_y F_q} = \frac{dq}{-(F_y + F_u u_y)} = \frac{dp}{-(F_x + F_u u_x)} \qquad (30)$$

It is convenient to re-write, here, the Charpit equations in terms of our variables and call $F := F_I(q_0, q, p_{0,}p, ; \gamma, \epsilon)$,independent of z, the Charpit functions corresponding to (17) and $F := F_{II}(q, p_{0,}p, z; \gamma)$, independent of q_0, those corresponding to (18 & 19),We have in both cases, recalling that $q_0 = c_0 t = s$,

$$\frac{dq_0}{ds} = F_{p_0} \qquad (31)$$

$$\frac{dp_0}{ds} = -(F_{q_0} + F_z p_0), \qquad (32)$$

$$\frac{dq}{ds} = F_{p,} \qquad (33)$$

$$\frac{dp}{ds} = -(F_q + F_z p), \qquad (34)$$

and

$$\frac{dz}{ds} = p_0 F_{p_0} + p F_p \qquad (35)$$

2.2 The Charpit functions for the present 1D fluids

With $z_{c_0 t} := p_0, c_0$ being the modell- dependant reference velocity defined in the last paragraph of sect.1. ,with $z_q := p$; dividing (17) by $c_S(y)$, and (18) , (19) by $c_T(T)^2$ and $c_S(y)^2$,respectively; setting $\gamma := g/c_T(T)^2$ or $= g/c_S(y)^2$, $v(q) : u(q)/c_T(T)$,or $u(q)/c_S(y)$, the Charpit functions F_I (corresponding to the ,only isentropic, Riemannian cases) and F_{II} (corresponding to the other,isothemal or isentropic, constant energy cases represented in colomn) , become, while re-calling that F_I depends upon ϵ but is independant of z,whereas F_{II} is independant of $q_o$and ϵ but dependant upon z :

$$F_I(q_0, q, p_0, p; \gamma, \epsilon) = p_0 + \epsilon(p^{-1} - 1) - v(q) + \gamma q_0 = 0 \qquad (36)$$

and

$$F_{II}(q, p_0, p, z; \gamma) \; = \frac{1}{2}p_0^2 + \begin{pmatrix} \ln p^{-1} \\ \frac{1}{2}(p^{-2} - 1) \end{pmatrix} - \frac{1}{2}v(q)^2 + \gamma(z - q) = 0. \qquad (37)$$

3 The gravitational Riemann invariants

We proceed with an analysis of the F_I models exclusivly in the following two Subsections, one devoted to their EDO's and to the equations for their densities, and one to examples of isentropic solutions.

3.1 EDO's of the Charpit functions (36) and equations for the densities

In writing the EDO's of the isentropic cases associated to the PDE $F_I(q_0, q, p_0, p; \gamma, \epsilon,)$ (36), we have

$$\frac{dq_0}{ds} = 1, \tag{38}$$

$$\frac{dp_0}{ds} = -\gamma, \tag{39}$$

$$\frac{dq}{ds} = -\epsilon p^{-2} \tag{40}$$

$$\frac{dp}{ds} = v(q)_q \tag{41}$$

In fact, from (36), we have also

$$p = (1 + \epsilon(v(q) - p_0 - \gamma q_0)^{-1}. \tag{42}$$

We have lastly

$$\frac{dz}{ds} = p_0 - \epsilon p^{-1}. \tag{43}$$

Observe,in passing, that the r.h.s of (43) assumes also the form $p_0 + p\frac{dq}{ds}$.

The solutions of $p_0(r, s)$ and $q_0(s)$ are, with $p_0(r, 0) = z_{c_0 t}(0) = v(r)$, for the two cases, and $q_0(s) = s$,

$$p_o(r, s) = v(r) - \gamma s. \tag{44}$$

Thus $p_0(r, s) + \gamma q_0(s) = v(r)$ and

$$p(q, r) = (1 + \epsilon(v(q) - v(r))^{-1} \tag{45}$$

We have next, from (40 & 42),

$$-\epsilon s = \int_r^q dq'(p(q', r)^2 = \int_r^q dq'(1 + \epsilon(v(q' - v(r))^{-2} \tag{46}$$

and, following (43)

$$z(q, r, s) = \int_0^s ds'(p_0(r, s') - \epsilon(q(s'), r)^{-p1}) = r + v(r)s - \frac{1}{2}\gamma s^2 + \epsilon \int_r^q dq' p(q'), r) \tag{47}$$

where, in the last equation, we have utilized the fact that,with $(eq. \ 40)$: $-ds' = \epsilon p(q', r)^2 dq'$.

We proceed with the equations for the densities. Following (15) and the convention of notation made at the end of Sect.1,we have

$$\rho(q.s) = \rho_0(q) \left(z_{q|s}\right)^{-1}. \tag{48}$$

It is convenient to use the following formulation:

$$z_q(q, r(q, s)) := z_{q|s} = \partial z/\partial q_{|r} + \partial z/\partial r_{|q} \partial r/\partial q_{q|s} := z_{q|r} + z_{r|q} r_{q|s} \tag{49}$$

and since $ds = s_q dq + s_r dr = 0$ we have

$$r_{q|s} = -\left(s_q/s_r\right)_s \tag{50}$$

and thus

$$z_{q|s} = z_{q|r} - z_{r|q} \left(s_q/s_r\right)_s \tag{51}$$

The ground is prepared for presenting exemples of solutions of our gravitational (isentropic) Riemann invariants.

3.2 Examples

Depending upon the initial conditions for the velocty fields and densities of the F_I models, the following examples are proposed for investigation**:** $(F_I\ , a)$: $\rho_0(r) = \rho_0$, $v(r) = r/c_0\tau = \lambda r$, $\kappa(r) = 1$; (F_I, b) : $\rho_0(r) = \rho_0$, $v(r)= (u_\infty/c_0) \tan h(\mu r)$, $u_\infty/c_0 := v_\infty$, $\mu v_\infty = \lambda, \kappa(r) = 1$ and (F_I, c) : $\rho_0(r) = \rho_0(Ch(\mu r))^{-2} = \rho_0(1 - \tanh(\mu r)^2$, $v(r)=v_\infty \tan h(\mu r) := v_\infty \theta(r)$, i.e $\kappa(r) := 1- \ \theta(r)^{2.}$ .Notice that the cases $(F_I a)$ and $(F_I c)$ are said to be correlated in the sense that their initial densities $\rho(r)$ are $\backsim \partial v(r)/\partial r$.

The general procedure is $i)$:to solve $p_0(r, s)$; $ii)$: to solve $q_0(r, s)$; $iii)$ to use the Chapit functions F_I to express p in function of q and r,in order to solve the parametric equation $s = s(q, r)$,and $iv)$: to solve the equation for $z(r, q, s)$.The next operations which, in general, require numerical analysis, are to invert $s(q, r) \to r(q, s)$, since q and s are pescribed, to feed the result in $z(q, r(q, s), s) := z(q, s)$ and in $\rho(q, s)$. Lastly,the independant variables q with $0 \le q < \infty$ and s in $\rho(q, s)$ and in $z(q, s)$ are re-labeled as $q \to y$,and , $s \to c_0 t$, thus generating the sougbt solutions of the original PDE' (17).

Recall that, here,we have, $F_I(q_0, q, p_0, p; \gamma, \epsilon,) = p_0 + \epsilon\left(p^{-1} - 1\right) - v(q) + \gamma q_0 = 0$, $\gamma = g/c_0^2$. Then,from (38 & 39), $dq_0/ds = 1; dp_0/ds = -\gamma; and\ dp/ds = v(q)_q$, (41); $dq/ds = -\epsilon p^{-2}$, (40) and $dz/ds = p_0 - \epsilon p^{-1} = p_0 + p dq/ds$, (43).

-Example $F_I, S, a)$: $\rho_0(q) = \rho_0, k(q) = 1, v(q) = \lambda q$.We have again $p_0 = v(r) - \gamma s$ and $q_0 = s$.Next, with $p_0 + \gamma q_0 = v(r)$ and (42) for p,we get

$$p^{-1} = 1 + \epsilon(v(q) - v(r)) \tag{52}$$

and,from (40)

$$-s = \epsilon \int_r^q dq' p(q'.r)^2 = \tag{53}$$

$$\epsilon \int_r^q dq'(1+\epsilon(v(q')-v(r))^{-2} = \epsilon \int_r^q dq'(1+\epsilon(\lambda q' - \lambda r))^{-2} \tag{54}$$

i.e.,

$$-s = \frac{1}{\lambda}(\frac{-1}{1+\epsilon\lambda(q-r)}+1). \tag{55}$$

This relation is easily invertible to yield

$$r(q,s) = q + \frac{\epsilon s}{1+\lambda s}. \tag{56}$$

We have next

$$z(q,r,s) \quad = \quad r + \lambda rs - \frac{1}{2}\gamma s^2 - \epsilon \int_0^s ds' p^{-1} = \tag{57}$$

$$r + \lambda rs - \frac{1}{2}\gamma s^2 + \int_r^q dq' p(q',r) \tag{58}$$

i.e.,

$$z(q,r,s) = r(1+\lambda s) - \frac{1}{2}\gamma s^2 + \epsilon\lambda^{-1}\ln \mid 1+\lambda\epsilon(q-r) \mid . \tag{59}$$

Inserting (56) for $r(q,s)$ gives

$$z(q,r(q,s),s) = z(q,s) = (1+\lambda s)\left(q+\frac{\epsilon s}{1+\lambda s}\right) - \frac{1}{2}\gamma s^2 - \epsilon\lambda^{-1}\ln \mid 1+\lambda s \mid . \tag{60}$$

Recovering the original notations $q \to y, s \to c_o t$ and remembering that $\lambda s = t/\tau$,gives

$$z(y,t) = (1+t/\tau)(y+\frac{\epsilon c_0 t}{1+t/\tau}) - \frac{1}{2}gt^2 - \epsilon c_0\tau \ln \mid 1+t/\tau \mid = \tag{61}$$

$$(1+t/\tau)\,(y+\epsilon c_0\tau) - \epsilon c_0\tau - \frac{1}{2}gt^2 - \epsilon c_0\tau\ln \mid 1+t/\tau \mid \tag{62}$$

It remains to check that (62) coincides with a solution to be found in [3], the book by Polyanin and Zaitzev. Indeed,this is achieved in three steps : $i)$ separate the gravitational factor gt from (17) in setting $z = \varsigma - \frac{1}{2}gt^2$, $ii)$ rewrite the EDP satisfied by ς, namely (17) without the term gt and mullti-ply it by $\varsigma_{y,}$,thus producing the PDE: $\varsigma_y\varsigma_t - (\epsilon c_0 + y/\tau)\varsigma_y + \epsilon c_0 = 0$, isomorphic to $[3.No.2.1, 2, 3.p.45]$, $iii)$ identify the correspondance between the notations of the general solution of [3] and that particular of (62),namely: $(w,x,y,a,b,c,s,C_1,C_2) \to (-\varsigma, y, t, 1/\tau, \epsilon c_0\tau, 0, \epsilon c_0, 1, \epsilon c_0)$.

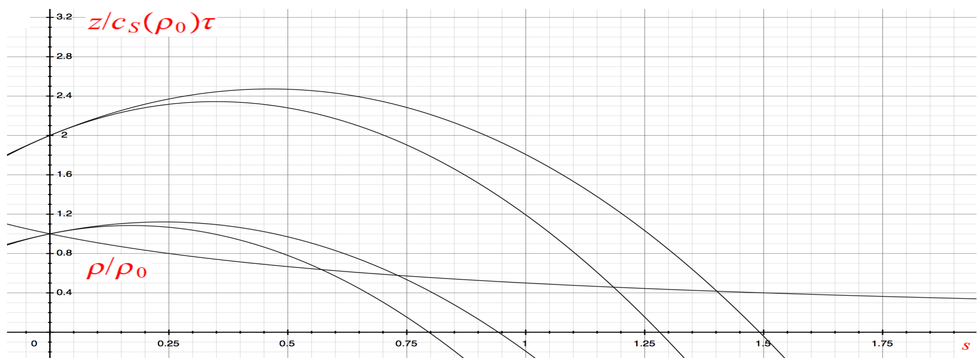

Figure 1: Characteristics (62) and density (63).

It remains to give the solution for the density.We have

$$\rho(q,s) = \rho_0 z_q^{-1} \to \rho(y,t) = \rho_0\ (1 + t/\tau), \tag{63}$$

a solution again valid in the time domain bounded by the requirement that $z(y,t) \geq 0$.

On Fig.1, first publishd in [5] , four examples of characteristcs (62) are presented and one,(63),for the density, the latter, plotted in units of ρ_0 ,with 1 as initial value. If $s = t/\tau$, if z and y are plotted in units of $c_S(\rho_0)\tau$, while g, in units of $c_s(\rho_0)/\tau)$, choosen to be $= 5$; then, the four characteristics shawn correspond to the initial values : $(\epsilon, y(c_S(\rho_0)\tau)^{-1}) := (-1,1;1,1;-1,2;1,2)$, and are presented from bottom to top, in the relevant first quadrant of the variables.

Example $F_I, S, b)$: $\rho_0(q) = \rho_0,\ \kappa(q) = 1,\ v(r) = v_\infty \tanh{(\mu r)}$ .Here, we have also $p_0 = v(r) - \gamma s$ and $q_0 = 1$. Next, with $(49dawn)$ and $k(q) = 1$, $p^{-1} = 1 + \epsilon v_\infty(\tanh{(\mu q)} - \tanh{(\mu r)})$,we have

$$z(q,r,s) = r + v(r)s - \frac{1}{2}\gamma s^2 - \epsilon \int_0^s ds' p(q(s',r)^{-1}. \tag{64}$$

0r,with $v(q) = v_\infty \tanh{(\mu q)}$,

$$z(q,r,s) = r + v(r)s - \frac{1}{2}\gamma s^2 \int_r^q dq'(1 + \epsilon v_\infty(\tanh{(\mu q')} - \tanh{(\mu r)})^{-1}. \tag{65}$$

Next, and with $dq/ds = -\epsilon p^{-2}$,we get

$$-s(q,r) = \epsilon \int_r^q dq' p\left(q',r\right)^2 = \epsilon \int_r^q dq'(1+\epsilon v_\infty(\tanh\left(\mu q'\right) - \tanh\left(\mu r\right))^{-2}, \quad (66)$$

Setting as before $\theta = \tanh(\mu q'), \theta_1 = \tanh \mu q, \theta_0 = \tanh(\mu r)$, $dp/ds = -\epsilon k(q)(p^{-1}-1) + v(q), (46dawn)$ and

$dq' = \mu^{-1}\left(1-\theta^2\right)^{-1} d\theta$, the above equations become

$$z(q,r,s) = r + v(r)s - \frac{1}{2}\gamma s^2 \mu^{-1} \int_{\theta_0}^{\theta_1} d\theta \left(1-\theta^2\right)^{-1}(1+\epsilon v_\infty(\theta-\theta_0)^{-1} \quad (67)$$

and

$$-\mu s(q,r) = \epsilon \int_{\theta_0}^{\theta_1} d\theta \left(1-\theta^2\right)^{-1}(1+\epsilon v_\infty(\theta-\theta_0)^{-2}, \quad (68)$$

both results being integrable, but remaining implicit functions of θ_0 and θ_1.

Exemple $F_I, S, c)$ $\rho_0(q) = \rho_0 k(q) = \rho_0 \cosh(\mu q)^{-2} = \rho_0(1-(\tanh \mu q)^2), v(q) = v_\infty \tanh(\mu q) = v_\infty \theta(\mu q), k(q) = 1-\theta(q)^2$.We have once more $= v(r) - \gamma s = v_\infty \tanh(\mu r) - \gamma s, q_0 = s$,and next

$$p(q,r) = (1+\epsilon v_\infty(\tanh(\mu q) - \tan h(\mu r))^{-1}, \quad (69)$$

$$-s = \epsilon \int_r^q dq' p(q',r)^2 \quad (70)$$

$$= \epsilon \int_r^q dq'(1+\epsilon v_\infty(\tanh(\mu q') - \tan h(\mu r))^{-2}, \quad (71)$$

$$z(q,r,s) = r + v(r)s - \frac{1}{2}\gamma s^2 - \int_r^q dq' p(q') = \quad (72)$$

$$q + v(r)s) - \frac{1}{2}\gamma s^2 - \int_r^q dq'(1+\epsilon v_\infty(\tanh(\mu q') - \tan h(\mu r))^{-1}, \quad (73)$$

and, as in $F_I, S, b)$,except for the initial density,

$$\rho(q,s) = \rho_0(\cosh(\mu q)^{-2}) p(q, r(q,s), s)^{-1}. \quad (74)$$

Notice here again, that, with $\theta = \tanh \mu q', d\theta = \mu(1-\tanh(\mu q')^2)dq'$,we have

$$-s = \mu^{-1}\ \epsilon \int_{\theta_0}^{\theta_1} d\theta(1-\theta^2 + \epsilon v_\infty(\theta - \theta(\mu r))^{-2} \quad (75)$$

and

$$z = r + v(r)s - \frac{1}{2}\gamma s^2 - \int_{\theta_0}^{\theta_1} d\theta(1-\theta^2+\epsilon v_\infty(\theta-\theta(\mu r))^{-1}. \quad (76)$$

Notice also that,again,both integrals can be peformed explicitely and if we define

$$L(\xi) := \int_{\theta_0}^{\theta_1} d\theta(\xi - \theta^2 + \epsilon v_\infty(\theta - \theta(\mu r))^{-1}, \tag{77}$$

then,

$$z = q + v(r)s - L(1), \tag{78}$$

and

$$\mu s = \frac{dL(\xi)}{d\xi} \mid \xi = 1. \tag{79}$$

In summary and for all the Riemann-isentropic cases, we hve : $p = 1/(1 + v(q) - v(r))$, $-s = \epsilon \int_r^q dq' k(q')^{-1} p(q', r)^2$,$z(q, r, s) = r + v(r)s - \frac{1}{2}\gamma s^2 + + \int_r^q dq' p(q', r)$.It follows that, $z_{q|r} = p$,$z_{r|q} = 1 + v(r)_r s - 1 + \int_r^q dq' p(q', r)_r = +v(r)_r s + v(r)_r \int_r^q dq' p(q', r)^2 = 0$.Thus, $z_{q|s} = p$ and,with $R.S.$ standing fo $Riemann - isentropic$ $\rho(q, s, \gamma; R, S) = \rho_0 k(q) p(r(q, s), s)^{-1}$, It is interesting to notice that the proportionality $\rho \sim p^{-1}$applies to all gravitational Riemannian cases.This is a noteworthy conclusion of the present work.

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