PHASE RETRIEVAL IN $\ell_2(\mathbb{R})$

SARA BOTELHO-ANDRADE, UNIVERSITY OF MISSOURI, SANDRADE102087@GMAIL.COM
PETER G. CASAZZA, UNIVERSITY OF MISSOURI, CASAZZAP@MISSOURI.EDU
DESAI CHENG, UNIVERSITY OF MISSOURI, CHENGDESAI@YAHOO.COM
JOHN HAAS, UNIVERSITY OF MISSOURI, TERRAFORMTHEDREAMSCAPE@GMAIL.COM
TIN T. TRAN, UNIVERSITY OF MISSOURI, TINMIZZOU@GMAIL.COM

Abstract. Real phase retrieval is one of the most active areas of research today. Although there are many papers on finite dimensional real phase retrieval, very little work has been done on the infinite dimensional case which is what we will address here. We will review the major results in finite dimensional real phase retrieval for vectors and projections and then see which results extend to infinite dimensions. In particular, we will:

(1) show that *Edidin's theorem* extends to infinite dimensions.
(2) show that there are vectors doing phase retrieval but their perps fail phase retrieval.
(3) show there is a family of vectors doing phase retrieval but if any vector is deleted, it fails phase retrieval.
(4) extend the notion of *full spark* to infinite dimensions and see that full spark families do phase retrieval.
(5) show that the families of vectors which do phase retrieval are not dense in the families of vectors in ℓ_2.
(6) show that there are families of three Riesz bases that do phase retrieval but leave open the question of where two Riesz bases can do phase retrieval.
(7) give several constructions of families of vectors that do phase retrieval in ℓ_2.

1. Introduction

Phase retrieval is one of the most applied and studied areas of research today. Phase retrieval for Hilbert space frames was introduced in [2] and quickly became an industry. Although much work has been done on the complex infinite dimensional case of phase retrieval, only one paper exists on infinite dimenional real phase retrieval [3]. Here we will review the major results on finite dimensional real phase retrieval and show:

(1) Which results hold in infinite dimensions;
(2) Which results fail in infinite dimensions;
(3) Which results are unknown in infinite dimensions.

We will need the definition of a Hilbert space frame.

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Definition 1.1. *A family of vectors* $\{x_i\}_{i\in I}$ *in a finite or infinite dimensional Hilbert space* $\mathbb{H}$ *is a* **frame** *if there are constants* $0 < A \leq B < \infty$ *so that*

$$A\|x\|^2 \leq \sum_{i\in I} |\langle x, x_i\rangle|^2 \leq B\|x\|^2, \text{ for all } x \in \mathbb{H}. \tag{1}$$

(1) If $A = B$ this is an A-**tight** frame.
(2) If $A = B = 1$, this is a Parseval frame.

We also need to work with Riesz sequences.

Definition 1.2. *A family* $X = \{x_i\}_{i\in I}$ *in a finite or infinite dimensional Hilbert space* $\mathbb{H}$ *is a* **Riesz sequence** *if there are constants* $0 < A \leq B < \infty$ *satisfying for all sequences of scalars* $\{a_i\}_{i\in I}$ *we have:*

$$A\sum_{i\in I} |a_i|^2 \leq \|\sum_{i\in I} a_i x_i\| \leq B\sum_{i\in I} |a_i|^2. \tag{2}$$

If the closed linear span of X *equals* $\mathbb{H}$*, we call* X *a* **Riesz basis***.*

The complement property and full spark will be a major tool here.

Definition 1.3. *A family of vectors* $\{x_i\}_{i=1}^{\infty}$ *in* ℓ_2 *has the* **complement property** *if for every* $I \subset \mathbb{N}$ *either* $\overline{\text{span}}_{i\in I} = \ell_2$ *or* $\overline{\text{span}}_{i\in I^c} = \ell_2$.

Definition 1.4. *A family of vectors* $\{x_i\}_{i=1}^{m}$ *in* $\mathbb{R}^n$ *is* **full spark** *if for every* $I \subset [m]$ *with* $|I| = n$ *we have that* $\{x_i\}_{i\in I}$ *is linearly independent (hence spans* $\mathbb{R}^n$*).*

Throughout the paper, $\{e_i\}_{i=1}^{\infty}$ will be used to denote the canonical orthonormal basis for the real Hilbert space ℓ_2.

2. Finite Dimensional Results Which Carry Over to Infinite Dimensions

In this section we look at finite dimensional real phase retrieval and norm retrieval results which carry over to infinite dimensions.

2.1. Phase Retrieval.

We start with the definitions.

Definition 2.1. *A family of vectors* $\{x_i\}_{i\in I}$ *in* ℓ_2 *does* **phase retrieval** *if whenever* $x, y \in \ell_2$ *satisfy*

$$|\langle x, x_i\rangle| = |\langle y, x_i\rangle|, \text{ for all } i = 1, 2, \ldots, \tag{3}$$

then $x = \pm y$.

A family of projections $\{P_i\}_{i\in I}$ *in* ℓ_2 *does* **phase retrieval** *if whenever* $x, y \in \ell_2$ *and*

$$\|P_i x\| = \|P_i y\|, \text{ for all } i = 1, 2, \ldots, \tag{4}$$

then $x = \pm y$.

The following result appeared in [3]. The corresponding finite dimensional result first appeared in [4].

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Theorem 2.2. *A family of vectors in real ℓ_2 does phase retrieval if and only if it has the complement property.*

It follows that results on phase retrieval in the finite dimensional case which just depend on the complement property will also hold in ℓ_2.

Theorem 2.3. *Let $X = \{x_i\}_{i=1}^\infty$ do phase retrieval.*

(1) *Then so does $\{a_i x_i\}_{i=1}^\infty$ for $a_i \neq 0$ for all i.*
(2) *If T is an invertible operator then $\{Tx_i\}_{i=1}^\infty$ does phase retrieval.*

The finite dimensional version of the next theorem first appeared in [7].

Theorem 2.4. *A family of projections $\{P_i\}_{i=1}^\infty$ on ℓ_2 does phase retrieval in ℓ_2 if and only if for every $0 \neq x \in \ell_2$, $\overline{\text{span}}\{P_i x\}_{i=1}^\infty = \ell_2$.*

Proof. ($\Rightarrow$) We proceed by way of contradiction. So assume that there is an $0 \neq x \in \ell_2$ and $\{P_i x\}_{i=1}^\infty$ does not span ℓ_2. Choose $0 \neq y \in \ell_2$ so that $y \perp P_i x$ for all $i = 1, 2, \ldots$. Let $u = x + y$ and $v = x - y$. Then since $P_i y \perp P_i x$ for all i, we have that

$$\|P_i(x+y)\|^2 = \|P_i x\|^2 + \|P_i y\|^2 = \|P_i(x-y)\|^2. \tag{5}$$

If $\{P_i\}_{i=1}^\infty$ does phase retrieval, then

$$x + y = \pm(x - y).$$

This implies $x = 0$ or $y = 0$, which is a contradiction.

($\Leftarrow$) The proof of Edidin's theorem in [5] works directly here. □

So results in finite dimensions which depend only on Edidin's theorem also hold in ℓ_2.

Corollary 2.5. *The following are equivalent for a family of projections $\{P_i\}_{i=1}^\infty$ on ℓ_2:*

(1) *$\{P_i\}_{i=1}^\infty$ fails phase retrieval.*
(2) *There are vectors $\|x\| = \|y\| = 1$ in ℓ_2 so that $P_i x \perp P_i y$ for all i.*

The finite dimensional version of the following theorem appeared in [4]. The proof in that case immediately works in ℓ_2.

Theorem 2.6. *Let $\{P_i\}_{i=1}^\infty$ do phase retrieval on ℓ_2.*
Then $\{(I - P_i)\}_{i=1}^\infty$ does phase retrieval if and only if it does norm retrieval.

In the finite dimensional case it is known [4] that there are projections $\{P_i\}_{i=1}^m$ which do phase retrieval but $\{(I - P_i)\}_{i=1}^m$ fails phase retrieval. We now give the infinite dimensional version of this

Theorem 2.7. *There is a set of vectors in ℓ_2 which do phase retrieval but their perps fail to do phase retrieval.*

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Proof. First, we will show that for each $n \in \mathbb{N}$, there exista a full spark set of $(2n-1)$ vectors $\{x_{nk}\}_{k=1}^{2n-1}$ in $\mathbb{R}^n$ such that the first coordinate of the vector $(I - P_{nk})(\varphi_n)$ equals zero, where

$$\varphi_n = \left(1, \frac{1}{2}, \dots, \frac{1}{n}\right) \tag{6}$$

and P_{nk} is the orthogonal projection onto x_{nk}.

Define the following subset of $\mathbb{R}^n$:

(7)
$$A_n := \left\{ \left(\sum_{i=1}^{n-2} t^{2i} + \left(n - n\sum_{i=2}^{n-1} \frac{t^{i-1}}{i} \right)^2, t, t^2, \dots t^{n-2}, n - n\sum_{i=2}^{n-1} \frac{t^{i-1}}{i} \right) : t \in \mathbb{R} \right\}.$$

Let any $x \in A_n$, and denote by P_x the projection onto x. Then

$$(I - P_x)(\varphi_n) = \varphi_n - \langle \varphi_n, \frac{x}{\|x\|} \rangle \frac{x}{\|x\|}. \tag{8}$$

Denote by a_1 and b_1 the first coordinate of x and $(I-P_x)(\varphi_n)$ respectively. Then we have

$$a_1 = \|x\|^2 - a_1^2,$$

and hence

$$b_1 = 1 - \frac{1}{\|x\|^2}(a_1 + 1)a_1 = 0.$$

Now we will show that for any finite family of hyperplanes in $\mathbb{R}^n$, there exists a point in A_n that does not lie in any of these hyperplanes, and therefore there exists a full spark of $(2n-1)$ vectors $\{x_{nk}\}_{k=1}^{2n-1} \subset A_n$.

Indeed, let $\{W_i\}_{i=1}^k$ be any finite set of hyperplanes in $\mathbb{R}^n$. Suppose, by way of contradiction, that $B_n \subset \cup_{i=1}^k W_i$. Then there exists $j \in \{1, \dots, k\}$ such that W_j contains infinitely many vectors in A_n.

Let $u = (u_1, u_2, \dots, u_n) \in W_j^\perp, u \neq 0$. Then we have

$$\langle u, x_t \rangle = 0$$

for infinitely many $x_t \in A_n$.

Hence,

(9)
$$u_1 \left(\sum_{i=1}^{n-2} t^{2i} + \left(n - n\sum_{i=2}^{n-1} \frac{t^{i-1}}{i} \right)^2 \right) + \sum_{i=1}^{n-2} u_{i+1} t^i + u_n \left(n - n\sum_{i=2}^{n-1} \frac{t^{i-1}}{i} \right) = 0$$

for infinitely many $t \in \mathbb{R}$.

This implies $u_1 = u_2 = \cdots = u_n = 0$, which is impossible.

Thus, we have shown that for each n, there exists a full spark set of $(2n-1)$ vectors $\{x_{k,n}\}_{k=1}^{2n-1}$ in $\mathbb{R}^n$ such that the first coordinate of the vector $(I - P_{nk})(\varphi_n)$ equals zero. Notice that $\{x_{nk}\}_{k=1}^{2n-1}$ does phase retrieval in $\mathbb{R}^n$.

Now, for each n, we consider x_{nk} as a vector in ℓ_2, where its j-coordinate is zero when $j > n$. Then the collection of all x_{nk}, $n \in \mathbb{N}, k = 1, 2, \dots, 2n-1$,

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do phase retrieval in ℓ_2. Indeed, suppose that $|\langle x, x_{nk}\rangle| = |\langle y, x_{nk}\rangle|$ for all n, k but $x \neq \pm y$. Then there is a n_0 such that

$$(x(1), x(2), \ldots, x(n_0)) \neq \pm(y(1), y(2), \ldots, y(n_0)). \tag{10}$$

But then the corresponding $\{x_{n_0 k}\}_{k=1}^{2n_0-1}$ does phase retrieval $\in \mathbb{R}^{n_0}$ which implies $(x(1), x(2), \ldots, x(n_0)) = \pm(y(1), y(2), \ldots, y(n_0))$, a contradiction.

Finally, we show that $\{x_{nk}^{\perp}\}_{n=1,k=1}^{\infty,\ \ 2n-1}$ fails phase retrieval in ℓ_2. We will use the same notation P_{nk} for the projection onto $x_{nk} \in \ell_2$, and let $\varphi = \{\frac{1}{n}\}_{n=1}^{\infty} \in \ell_2$. Then by our construction, the first coordinate of $(I - P_{nk})(\varphi)$ equals zero for all k, n. Therefore, $\overline{\text{span}}\{(I - P_{x_{nk}})(\varphi)\}_{nk} \neq \ell_2$. Therefore $\{x_{nk}^{\perp}\}_{n=1,k=1}^{\infty,\ \ 2n-1}$ fails phase retrieval by Theorem 2.4.

□

In finite dimensions it is known [2] that any family of vectors doing phase retrieval must contain at least $(2n - 1)$-vectors. It follows that a full spark family of vectors $\{x_i\}_{i=1}^{2n-1}$ does phase retrieval (since it has complement property) but if we delete any vector it fails phase retrieval. Now we give a construction to show that this result holds in infinite dimensions. The following example shows that there is a family of vectors in ℓ_2 which does phase retrieval but we cannot drop any vector and maintain phase retrieval. This also contains a new construction for frames doing phase retrieval.

First, we need the following lemma:

Lemma 2.8. *Let $\{e_i\}_{i=1}^{\infty}$ be the canonical orthonormal basis for ℓ_2. For any fixed i, if x is orthogonal to $e_i + e_j$ for infinitely many $j > i$, then $\langle x, e_i\rangle = \langle x, e_j\rangle = 0$ for all such j.*

Proof. Let $K = \{j : j > i, \langle x, e_i + e_j\rangle = 0\}$, then by assumption, the cardinality of K is infinite.

It is clear that $|\langle x, e_i\rangle| = |\langle x, e_j\rangle|$ for all $j \in K$. Suppose by a contradition that $|\langle x, e_j\rangle| > 0$ for all $j \in K$. Then we have

$$\|x\|^2 \geq \sum_{j \in K} |\langle x, e_j\rangle|^2 = \infty. \tag{11}$$

a contradiction. □

Example 2.9. *Let the family of vectors $X = \{e_i + e_j\}_{i<j}$. Then X does phase retrieval in ℓ_2 but we cannot drop any vector of X and maintain phase retrieval.*

Proof. Let I be any subset of the set $\{(i, j) : i < j\}$, and we can assume that $(1, j) \in I$ for infinitely many j. We will show that either $\{e_i + e_j\}_{(i,j)\in I}$ or $\{e_i + e_j\}_{(i,j)\in I^c}$ spans ℓ_2. Suppose $\{e_i + e_j\}_{(i,j)\in I}$ does not span ℓ_2. We will show that $\{e_i + e_j\}_{(i,j)\in I^c}$ spans ℓ_2.

Let any $x = (x(1), x(2), \ldots)$ be such that $\langle x, e_i + e_j\rangle = 0$ for all $(i, j) \in I^c$.

By assumption, there is $y = (y(1), y(2), \ldots), y \neq 0$ and $\langle y, e_i + e_j\rangle = 0$ for all $(i, j) \in I$. Let s be the smallest number such that $y(s) \neq 0$. By Lemma

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2.8, $(s,j) \notin I$ for infinitely many $j > s$. Hence there is $t > s$ such that $(s,j) \in I^c$ for all $j \geq t$. Again, by Lemma 2.8, we get

$$x(s) = x(j) = 0 \text{ for all } j \geq t. \tag{12}$$

We will now show that $x(j) = 0$ for all $j = 1, 2 \ldots t-1$. Suppose there is $1 \leq j < s$ such that $x(j) \neq 0$. This implies $(j,s) \notin I^c$. Thus $(j,s) \in I$ and hence $y(j) \neq 0$. But this contradicts the way we chose s. So $x(j) = 0$ for all $1 \leq j < s$.

Now let any $s < j < t$. If $(s,j) \in I^c$, then $x(j) = 0$. If $(s,j) \in I$, then $y(j) \neq 0$. Note that by assumption, $(1,j) \in I$ for infinitely many j, and hence by Lemma 2.8, we get that $y(1) = 0$. Thus, $(1,j) \notin I$. Therefore $(1,j) \in I^c$ and so $x(j) = x(1) = 0$. This completes the proof that $\{e_i + e_j\}_{(i,j)\in I^c}$ span ℓ_2.

Now we will show that we cannot drop any vector of X and maintain phase retrieval.

Fix any $(k,\ell), k < \ell$. Consider $Y = \{e_i + e_j : i < j, (i,j) \neq (k,\ell)\}$. Let

$$x = e_k + e_\ell,\ y = e_k - e_\ell. \tag{13}$$

Clearly, $x \neq \pm y$. For any vector $e_i + e_j \in Y$, we compute:

$$\langle x, e_i + e_j\rangle = \langle e_k, e_i\rangle + \langle e_k, e_j\rangle + \langle e_\ell, e_i\rangle + \langle e_\ell, e_j\rangle, \tag{14}$$

$$\langle y, e_i + e_j\rangle = \langle e_k, e_i\rangle + \langle e_k, e_j\rangle - \langle e_\ell, e_i\rangle - \langle e_\ell, e_j\rangle. \tag{15}$$

If $i = k$ then $j \neq \ell$, $i < \ell$ and $k < j$. Thus

$$\langle x, e_i + e_j\rangle = \langle y, e_i + e_j\rangle = 1. \tag{16}$$

If $j = k$, then $i < j = k < \ell$. So

$$\langle x, e_i + e_j\rangle = \langle y, e_i + e_j\rangle = 1. \tag{17}$$

Consider the case $i, j \neq k$. If $i = \ell$ then $j \neq \ell$. Hence

$$\langle x, e_i + e_j\rangle = 1, \text{ and } \langle y, e_i + e_j\rangle = -1. \tag{18}$$

If $i \neq \ell$ and $j = \ell$ then

$$\langle x, e_i + e_j\rangle = 1, \text{ and } \langle y, e_i + e_j\rangle = -1. \tag{19}$$

Finally, if $i \neq \ell$ and $j \neq \ell$ then

$$\langle x, e_i + e_j\rangle = \langle y, e_i + e_j\rangle = 0. \tag{20}$$

Thus, in all cases, we always have that

$$|\langle x, e_i + e_j\rangle| = |\langle y, e_i + e_j\rangle|, \text{ for all } e_i + e_j \in Y. \tag{21}$$

Since $x \neq \pm y$, Y cannot do phase retrieval.

□

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2.2. **Full Spark.**

Remark 2.10. It is known that the full spark families of vectors in $\mathbb{R}^n$ are dense in $\mathbb{R}^n$ in the sense that given any family of vectors $x = \{x_i\}_{i=1}^m$ in $\mathbb{R}^n$ and any $\epsilon > 0$, there is a full spark family of vectors $Y = \{y_i\}_{i=1}^m$ in $\mathbb{R}^n$ so that

$$d(x,y)^2 = \sum_{i=1}^{m} \|x_i - y_i\|^2 < \epsilon. \tag{22}$$

One interpretation of the definition of full spark is that any minimal number of vectors in the set which could possibly span, must span, i.e. any subset of n-vectors must span. The corresponding statement for ℓ_2 is:

Definition 2.11. *A family of vectors $\{x_i\}_{i=1}^\infty$ is* **full spark** *in ℓ_2 if every infinite subset spans ℓ_2.*

A full spark set clearly has complement property and hence does phase retrieval in the infinite dimensional case.

Theorem 2.12. *There exist full spark families of vectors in ℓ_2 which then do phase retrieval.*

Proof. Such an example can be found in Theorem 2 of [10].

There is another simpler way to do this using the following argument. Instead of ℓ_2 consider $L_2[0,1]$. It is known that if a sequence $a_n \neq a$ of numbers (real or complex) tends to a when $n \to \infty$, then the sequence of functions $f_n(t) = e^{a_n t}$ spans $L_2[0,1]$ (this is a standard application of the Hahn-Banach theorem together with the uniqueness theorem for holomorphic functions, see more in Appendix III of [8].) Since every subsequence of a_n also has the same limit, every subsequence of f_n also spans $L_2[0,1]$. □

2.3. **Norm Retrieval.**

Definition 2.13. *A family of projections $\{P_i\}_{i\in I}$ (I is finite or infinite) does* **norm retrieval** *in $\mathbb{H}$ if for any $x, y \in \mathbb{H}$ we have:*

$$\|P_i x\| = \|P_i y\| \text{ for all } i \in I \text{ then } \|x\| = \|y\|. \tag{23}$$

The finite dimensional version of the next theorem first appeared in [5].

Theorem 2.14. *A family of projections $\{P_i\}_{i=1}^\infty$ on ℓ_2 does norm retrieval if and only if for any $x \in \ell_2$, $x \in \overline{\text{span}}\{P_i x\}_{i=1}^\infty$.*

Proof. ($\Rightarrow$) Let any $x \in \ell_2$ and $y \perp \overline{\text{span}}\{P_i x\}_{i=1}^\infty$. Then

$$\langle P_i x, P_i y\rangle = \langle P_i x, y\rangle = 0, \text{ for all } i. \tag{24}$$

Let $u = x + y, v = x - y$ then we have $\|P_i u\| = \|P_i v\|$ for all i. Hence $\|u\| = \|v\|$. Note that $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(u-v)$. Now we compute:

$$\langle x, y\rangle = \frac{1}{4}\langle u+v, u-v\rangle = \frac{1}{4}(\|u\|^2 - \|v\|^2) = 0. \tag{25}$$

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Thus, $(\overline{\text{span}}\{P_i x\}_{i=1}^{\infty})^{\perp} \subset x^{\perp}$, or equivalently, $x \in \overline{\text{span}}\{P_i x\}_{i=1}^{\infty}$.

($\Leftarrow$) Suppose $x, y \in \ell_2$, and $\|P_i x\| = \|P_i y\|$ for all $i = 1, 2, \ldots$. Set

$$u = x + y, v = x - y. \tag{26}$$

Then we have $\langle P_i u, P_i v\rangle = 0$ for all i. Hence $u \perp \overline{\text{span}}\{P_i v\}_{i=1}^{\infty}$. Since $v \in \overline{\text{span}}\{P_i v\}_{i=1}^{\infty}$ then $u \perp v$. It follows that $\|x\| = \|y\|$. □

The finite dimensional version of the next result first appeared in [6].

Theorem 2.15. *A family of vectors $\{x_i\}_{i=1}^{\infty}$ does norm retrieval in ℓ_2 if and only if for every $I \subset \mathbb{N}$ if $x \perp \overline{\text{span}}\{x_i\}_{i\in I}$ and $y \perp \overline{\text{span}}\{x_i\}_{i\in I^c}$ then $x \perp y$.*

Proof. ($\Rightarrow$) We may assume that $\|x\| = \|y\| = 1$. Let $u = x + y, v = x - y$. Then

$$|\langle u, x_i\rangle| = |\langle v, x_i\rangle|, \text{ for all } i. \tag{27}$$

Since $\{x_i\}_{i=1}^{\infty}$ does norm retrieval, $\|u\| = \|v\|$. It follows that $x \perp y$.

($\Leftarrow$) Suppose $|\langle x, x_i\rangle| = |\langle y, x_i\rangle|$, for all i. Denote

$$I = \{i : \langle x, x_i\rangle = -\langle y, x_i\rangle\}. \tag{28}$$

Then

$$I^c = \{i : \langle x, x_i\rangle = \langle y, x_i\rangle\}. \tag{29}$$

Let $u = x + y, v = x - y$. Then $u \perp \overline{\text{span}}\{x_i\}_{i\in I}$ and $v \perp \overline{\text{span}}\{x_i\}_{i\in I^c}$. Therefore, $u \perp v$, and we get $\|x\| = \|y\|$.

□

Corollary 2.16. *If $\{x_i\}_{i=1}^{\infty}$ is a Parseval frame in ℓ_2, let for $I \subset \mathbb{N}$, let*

$$H_1 = \overline{\text{span}}\{x_i\}_{i\in I} \text{ and } H_2 = \overline{\text{span}}\{x_i\}_{i\in I^c}, \tag{30}$$

then $H_1 \perp H_2$.

3. Finite Dimensional Results Which Fail in Infinite Dimensions

It is known [4] that the families of vectors $\{x_i\}_{i=1}^{m}$ which do phase retrieval in $\mathbb{R}^n$ are dense in the family of $m \geq (2n-1)$-element sets of vectors in $\mathbb{R}^n$. This follows from the fact that full spark families of $m \geq 2n - 1$ vectors are dense and do phase retrieval. The corresponding result fails in infinite dimensions.

Definition 3.1. *We say a family of sequences of vectors $\mathcal{F}$ is* **dense** *in ℓ_2 if given any sequence of vectors $Y = \{y_i\}_{i=1}^{\infty} \subset \ell_2$ and any $\epsilon > 0$ there an $X = \{x_i\}_{i=1}^{\infty} \in \mathcal{F}$ so that*

$$d(X, Y)^2 = \sum_{i=1}^{\infty} \|x_i - y_i\|^2 < \epsilon.$$

Remark 3.2. Note that a Riesz basis cannot do phase retrieval since it clearly fails complement property.

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Proposition 3.3. *Let* $X = \{x_i\}_{i=1}^{\infty} \subset \ell_2$ *be such that*

$$\sum_{i=1}^{\infty} \|x_i - e_i\|^2 \le 1 - \epsilon. \tag{31}$$

Then X *is a Riesz basis for* ℓ_2.

Proof. Define an operator $T : \ell_2 \to \ell_2$ by $Te_i = x_i$, for all $i = 1, 2, \ldots$. Given $a = \sum_{i=1}^{\infty} a_i e_i \in \ell_2$ we have

$$\|(I-T)a\|^2 = \|\sum_{i=1}^{\infty} a_i(e_i - x_i)\|^2 \tag{32}$$

$$\le \sum_{i=1}^{\infty} |a_i|\|e_i - x_i\| \tag{33}$$

$$\le \left(\sum_{i=1}^{\infty} |a_i|^2\right)\left(\sum_{i=1}^{\infty} \|x_i - e_i\|^2\right) \tag{34}$$

$$\le (1-\epsilon)\|a\|^2. \tag{35}$$

It follows that T is an invertible operator and so $\{Te_i\}_{i=1}^{\infty}$ is a Riesz basis for ℓ_2. □

Proposition 3.4. *The families of vectors which do phase retrieval in* ℓ_2 *are not dense in the infinite families of vectors in* ℓ_2.

Proof. Let $0 < \epsilon < 1$. If $X = \{x_i\}_{i=1}^{\infty}$ is any family of unit vectors with

$$\sum_{i=1}^{\infty} \|x_i - y_i\|^2 < 1 - \epsilon, \tag{36}$$

then X is a Riesz basis and hence cannot do phase retrieval. □

It is known in finite dimensions [1, 4] that if $X = \{x_i\}_{i=1}^{m}$ does phase retrieval, there is an $\epsilon > 0$ so that whenever $Y = \{y_i\}_{i=1}^{m}$ satisfies:

$$\sum_{i=1}^{m} \|x_i - y_i\|^2 < \epsilon, \tag{37}$$

then Y does phase retrieval. The above is called a **ϵ-perturbation** of X. The corresponding result fails in ℓ_2 as was shown in [3].

Theorem 3.5. *Given a frame* $\{x_i\}_{i=1}^{\infty}$ *doing phase retrieval in* ℓ_2 *and an* $\epsilon > 0$, *there is a frame* $\{y_i\}_{i=1}^{\infty}$ *which fails phase retrieval in* ℓ_2 *and satisfies:*

$$\sum_{i=1}^{\infty} \|x_i - y_i\|^2 < \epsilon. \tag{38}$$

Definition 3.6. *A set of vectors* $\{x_i\}_{i=1}^{\infty}$ *in* ℓ_2 *is* **finitely full spark** *if for every* $I \subset \mathbb{N}$ *with* $|I| = n$, $\{P_I x_i\}_{i=1}^{\infty}$ *is full spark (i.e. spark n+1), where* P_I *is the orthogonal projection onto* $\text{span}\{e_i\}_{i\in I}$.

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We will have to generalize the definition of full spark.

Definition 3.7. *A set of vectors* $\{x_i\}_{i=1}^m$ *in* $\mathbb{R}^n$ *is full spark if either they are independent or if* $m \geq n+1$*, then they have spark* $n+1$*.*

Proposition 3.8. *The finitely full spark families of vectors in* ℓ_2 *are dense in the infinite families of vectors in* ℓ_2*. In particular, there are Riesz bases for* ℓ_2 *which are finitely full spark, and these families cannot do phase retrieval.*

Proof. Let $\{y_i\}_{i=1}^\infty$ be a family of vectors in ℓ_2 and fix $\epsilon > 0$. We will construct the vectors by induction. To get started, choose a vector x_1 with all non-zero coordinates so that $\|x_1 - y_1\|^2 < \frac{\epsilon}{2}$. Now assume we have constructed vectors $\{x_i\}_{i=1}^m$ so that for every $I \subset \mathbb{N}$ with $|I| < \infty$, $\{P_I x_i\}_{i=1}^m$ is full spark and $\|x_i - y_i\|^2 < \frac{\epsilon}{2^{i+1}}$. For each finite subset $I \subset \mathbb{N}$, let

(39)

$$\mathcal{G}_I = \bigcup \left\{ \overline{\text{span}} \left[\{P_I x_i\}_{i \in I'} \cup \{e_i\}_{i \in I^c} \right] : I' \subset [m], \begin{cases} |I'| = m \text{ if } m+1 \leq |I| \\ |I'| = |I| - 1 \text{ if } |I| \leq m \end{cases} \right\}.$$

Let

$$\mathcal{F} = \bigcup_{n=1}^\infty \bigcup_{|I|=n} \mathcal{G}_I, \tag{40}$$

then $\mathcal{F}$ is a countable union of proper subspaces of ℓ_2 and hence there exists a vector y_{m+1} not in $\mathcal{F}$ and $\|x_{m+1} - y_{m+1}\|^2 < \frac{\epsilon}{2^{m+1}}$. This provides the required family of finitely full spark vectors. □

4. Lifting

In this section we demonstrate an embedding of finite frames in higher dimensions such that the complement property is preserved, which we will refer to as "lifting". We provide necessary and sufficient conditions for when such a construction is possible and an example to demonstrate problems that may arise in infinite dimensions. We begin with a few useful definitions.

Definition 4.1. *A frame* $X = \{x_i\}_{i \in I}$ *has the overcomplete complement property if for every* $S \subset I$*, either* $\{x_i\}_{i \in S}$ *or* $\{x_i\}_{i \in S^c}$ *spans and is linearly dependent, i.e. it is not a basis.*

The overcomplete complement property is a natural generalization of the usual complement property, as will be shown shortly. Next we specify exactly what types of embeddings we are considering.

Definition 4.2. *A frame* $Y = \{y_i\}_{i=1}^m \subset \mathbb{R}^{n+k}$ *is a* **k-lifting** *of a frame* $\{x_i\}_{i=1}^m$ *if*

$$y_i|_{\mathbb{R}^n} = x_i, \text{ for all } i = 1, 2, \ldots, m. \tag{41}$$

The next theorem classifies when 1-lifts are possible and provides a construction for the choice of coordinates to adjoin.

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Theorem 4.3. *A phase retrievable frame $X = \{x_i\}_{i=1}^m \subset \mathbb{R}^n$ can be* 1*-lifted to a phase retrievable frame if and only if X has the overcomplete complement property.*

Proof. For the sufficiency we shall provide a constructive proof. The idea of the proof will be to produce a vector $v \in \mathbb{R}^m$ such that the i^{th} coordinate of v will be the $(n+1)^{th}$ coordinate of $\hat{x}_i$. Given a subset $S \subset [m]$, by assumption either $X_S = \{x_i\}_{i\in S}$ or $X_{S^c} = \{x_i\}_{i\in S^c}$ spans $\mathbb{R}^n$ and is linearly dependent. We begin by demonstrating an embedding of vectors from the spanning set that still span in $\mathbb{R}^{n+1}$. Without loss of generality, in our notation we shall assume X_S is always the overcomplete spanning set of vectors. Then for some choice of coefficients we have $\sum_{i\in S} \alpha_i x_i = 0$ where α_i are not all zero. Denote $\alpha_S = (\alpha_1, \alpha_2, ..., \alpha_{|S|}) \in \mathbb{R}^{|S|}$ and pick $\beta_S \in \mathbb{R}^{|S|}$ such that $\langle \alpha_S, \beta_S \rangle \neq 0$. Define the embedded vectors $\hat{X}_S = \{\hat{x}_i\}_{i\in S} \in \mathbb{R}^{n+1}$ as follows

$$\hat{x}_i(j) = \begin{cases} x_i(j) & j \in [n] \\ \beta_S(i) & j = n+1. \end{cases} \tag{42}$$

To show that $\hat{X}_S$ spans $\mathbb{R}^{n+1}$, observe $\frac{1}{\langle \alpha_S, \beta_S \rangle} \sum_{i\in S} \alpha_i \hat{x}_i = e_{n+1}$. Since X_S spans $\mathbb{R}^n$, it follows that $\hat{X}_S$ spans $\mathbb{R}^{n+1}$.
This construction gives a procedure for an embedding which spans the larger space $\mathbb{R}^{n+1}$, but is dependent on the subset S. Also observe we haven't posed any conditions on how to extend the vectors in S^c. For each choice of S, we have the associated vectors $\alpha_S, \beta_S \in \mathbb{R}^{|S|}$. Let $H_S \subset \mathbb{R}^{|S|}$ denote the hyperplane perpendicular to α_S. Then our construction depends on being able to choose a vector in the complement of H_S for all subsets S. But the cardinality of S is changing as we range over all possibilities. To overcome this we will work with the larger space $\mathbb{R}^m = \mathbb{R}^{|S|} \times \mathbb{R}^{|S^c|}$. There are finitely many choices of S therefore $\bigcup_{S\subset[m]} H_S \times \mathbb{R}^{|S^c|} \neq \mathbb{R}^m$. Then for $v \in \left(\bigcup_{S\subset[m]} H_S \times \mathbb{R}^{|S^c|}\right)^c$ we defined

$$\hat{x}_i(j) = \begin{cases} x_i(j) & j \in [n] \\ v(i) & j = n+1. \end{cases} \tag{43}$$

Then it follows that $\hat{X} = \{\hat{x}_i\}_{i=1}^m$ has the complement property in $\mathbb{R}^{n+1}$.

For necessity assume X does phase retrieval but does not have the overcomplete complement property. Any spanning set that is a basis cannot be 1-lifted since there will not be enough vectors to span $\mathbb{R}^{n+1}$. □

The result above may be generalized for a k-lift with minimal effort. Naturally the overcompleteness of each subset S is critical in determining what integers k are plausible. More specifically, we define the lifting number of a phase retrievable frame as follows:

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Definition 4.4. *Given a frame $X = \{x_i\}_{i\in[m]} \subset \mathbb{R}^n$, let*

$$L_X = \min\{|S| - n : \text{span}\{x_i\}_{i\in S} = \mathbb{R}^n \text{ and } |S| \geq |S^c|\}, \tag{44}$$

then L_X is the lifting number for the frame X.

From the previous theorem we see immediately that if Φ has the overcomplete complement property then $L_X \geq 1$. The lifting number tells us how many dimensions higher we can lift Φ. If $L_X \geq 1$ then when we 1-lift, each overcomplete spanning subset will be lifted to a spanning set in $\mathbb{R}^{n+1}$ with the same cardinality. If $L_X > 1$ that means each spanning subset with higher cardinality (S or S^c) will be lifted to a spanning set which is still not a basis in $\mathbb{R}^{n+1}$, hence can be lifted again. The idea is that after each lift, the lifting number of the subsequent lifted frame $\hat{\Phi}$ is one smaller than L_X. That is, if $\hat{X}$ is a 1-lift of X then $L_{\hat{X}} = L_X - 1$.

Corollary 4.5. *$X \subset \mathbb{R}^n$ can be k-lifted if and only if $k \leq L_X$.*

Theorem 4.6. *If a frame $X \subset \mathbb{R}^n$ contains $2n + 2m + 1$ vectors with a $2n + 2m$ full spark subset. X can be $(m + 1)$-lifted.*

Proof. Clearly if a frame contains a $2n + 2m$ full spark subset it does phase retrieval as it contains a $2n - 1$ full spark subset which already does phase retrieval. Let $X = \{x_i\}_{i\in[2n+2m+1]}$ and $H = \{x_i\}_{i\in[2n+2m]}$ be a full spark subset. Given any $S \subset [2n + 2m]$, if S contains more than half the elements in H then it will be a spanning set with more than $n + m$ vectors hence its cardinality minus n will be greater than m hence at least $m + 1$. If it contains less than half of the elements of H then the same holds for S^c. If it contained exactly half then both S and S^c will contain $n + m$ elements of H hence they both span. Whichever set that contains $x_{2n+2m+1}$ will be a spanning set of cardinality $n + m + 1$. Hence X will have lifting number $m + 1$. ☐

The set of $2n + 2m + 1$ full spark vectors is open, dense, and contains a subset of $2n + 2m$ full spark vectors. Then the previous theorem shows the set of $2n + 2m + 1$ vectors in $\mathbb{R}^n$ that can be $m + 1$-lifted contains an open dense set. Hence "almost every" set of $2n + 2m + 1$ or $2n + 2m + 2$ vectors can be $m + 1$-lifted.

The infinite dimensional version of this looks like the following. Note that this is not a classification of the liftable phase retrieving frames but a sufficient condition.

Remark 4.7. In ℓ_2, by a lift we "add" a coordinate at the beginning of the vector. That is, if $\hat{x}$ is a lift of x then $P\hat{x} = (0, x(1), x(2), \ldots)$, where P is the orthogonal projection onto $e_1^{\perp}$.

Theorem 4.8. *Let $X = \{x_i\}_{i=1}^{\infty}$ be a frame for ℓ_2 doing phase retrieval and let $Y = \{y_i\}_{i=1}^{\infty}$ be a linearly dependent spanning set in ℓ_2. Then $X \cup Y$ can be lifted to a phase retrieving frame for ℓ_2.*

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Proof. Let $X = \{x_i\}_{i=1}^\infty$ and $Y = \{y_i\}_{i=1}^\infty$ be as in the theorem. We show that we can lift this union to one higher dimension and maintain phase retrieval. Let L be the right shift operator on ℓ_2, i.e. if $x = (x(1), x(2), \ldots)$, then $Lx = (0, x(1), x(2), \ldots)$. Replace vectors in X by $\hat{X} = \{\hat{x}_i\}_{i=1}^\infty$ where $\hat{x}_i = Lx_i$. The idea for $\hat{Y} = \{\hat{y}_i\}_{i=1}^\infty$ is very similar to the proof in Theorem 4.3. We show existence of a vector $v = (v(1), v(2), \ldots) \in \ell_2$ such that $\hat{y}_i = v(i)e_1 + Ly_i$ will have the desired property to assure $\hat{X} \cup \hat{Y}$ does phase retrieval.

Since $\{y_i\}_{i=1}^\infty$ is linearly dependent, there exists a sequence of scalars $\alpha = \{\alpha_i\}_{i=1}^\infty$ with all but a finite number equal to zero, such that $\sum_{i=1}^\infty \alpha_i y_i = 0$. Denote $H_i = e_i^\perp \subset \ell_2$ and note that by the Baire Category Theorem

$$\left[(\cup_{i=1}^\infty H_i) \cup \alpha^\perp\right]^c \neq \emptyset. \tag{45}$$

Let $v \in \left[(\cup_{i=1}^\infty H_i) \cup \alpha^\perp\right]^c$ and define $\hat{y}_i$ as stated above. Note that v has all non-zero coordinates and $\langle v, \alpha \rangle \neq 0$. Moreover,

$$\sum_{i=1}^\infty \alpha_i \hat{y}_i = \sum_{i=1}^\infty (\alpha_i (v(i)e_1 + Ly_i) = \langle \alpha, v \rangle e_1. \tag{46}$$

Let any $j \geq 1$ and $\epsilon > 0$. Since $\{y_i\}_{i=1}^\infty$ spans ℓ_2, there is a finite subset I_j and scalars $\{\beta_k\}_{k \in I_j}$ such that

$$\|e_j - \sum_{k \in I_j} \beta_k y_k\| < \epsilon. \tag{47}$$

This implies

$$\|e_{j+1} - \sum_{k \in I_j} \beta_k (\hat{y}_k - v(k)e_1)\| < \epsilon, \text{ for all } j \geq 1. \tag{48}$$

Since $e_1 \in \text{span}\{\hat{y}_i\}_{i=1}^\infty$, $e_j \in \overline{\text{span}}\{\hat{y}_i\}_{i=1}^\infty$ for all j, $\hat{Y} = \{\hat{y}_i\}_{i=1}^\infty$ spans ℓ_2.

Now we will show that $\hat{X} \cup \hat{Y}$ satisfies Edidin's theorem. Since $\langle e_1, \hat{y}_i \rangle = v(i) \neq 0$, the projection of e_1 on the vectors of $\hat{Y}$ span ℓ_2. Let any non-zero vector $x \neq e_1$, the projection of x onto the $\hat{x}_i$'s will spans $e_1^\perp \subset \ell_2$. Note that x cannot be orthogonal to all $\hat{y}_i$ since these vectors span ℓ_2. Let $\hat{y}_j$ be one such vector. Since $\hat{y}_j$ is outside of $e_1^\perp \subset \ell_2$, the projection of x onto the vectors of $\hat{X} \cup \hat{Y}$ span ℓ_2 as well. Hence $\hat{X} \cup \hat{Y}$ does phase retrieval. □

5. Finite Dimensional Results which are not Known in Infinite Dimensions

It is known [4] that there are two orthonormal bases for $\mathbb{R}^n$ which do phase retrieval. We do not know the same for ℓ_2.

Problem 5.1.
Are there two Riesz bases for ℓ_2 which do phase retrieval?

However, we can do phase retrieval with three Riesz sequences for ℓ_2.

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Proposition 5.2. *There are three Riesz sequences for ℓ_2 which do phase retrieval.*

Proof. For every n let $H_n = \text{span}\{e_i\}_{i=1}^{3^{n+1}}$. Choose $\{u_{nij}\}_{i=3^n+1,j=1}^{3^{n+1},3}$ so that they are full spark in H_n and

$$\sum_{i=3^n+1}^{3^{n+1}} \|u_{nij} - e_i\|^2 \le \frac{1}{2^{n+1}} \quad \text{for} \quad j = 1, 2, 3 \tag{49}$$

Then the collection $\{u_{nij} : 3^n + 1 \le i \le 3^{n+1}, 1 \le j \le 3, 1 \le n < \infty\}$ does phase retrieval. Moreover,

$$\sum_{n=1}^{\infty} \sum_{i=3^n+1}^{3^{n+1}} \|u_{nij} - e_i\|^2 \le \frac{1}{2} \quad \text{for} \quad j = 1, 2, 3, \tag{50}$$

therefore $\{u_{nij}\}_{n=1,i=3^{n+1}}^{\infty,\ 3^{n+1}}$ is a Riesz sequence for $j = 1, 2, 3$. □

6. Sets Which do Phase Retrieval in ℓ_2

Theorem 6.1. *Assume we have subspaces $W_1 \subset W_2 \subset \cdots \subset \ell_2$ and vectors $\{x_{ij}\}_{j\in I_i}$ doing phase retrieval in W_i for every i. Finally, assume $\cup_{i=1}^{\infty} W_i$ is dense in ℓ_2. Then $\{x_{ij}\}_{i=1,j\in I_i}^{\infty}$ does phase retrieval in ℓ_2.*

Proof. We will check the complement property. Observe that a partition of vectors $\{x_{ij}\}_{i=1,j\in I_i}^{\infty}$ induces a partition for vectors $\{x_{ij}\}_{j\in I_i} \subset W_i$. By assumption $\{x_{ij}\}_{j\in I_i}$ does phase retrieval on W_i, therefore for each $i = 1, 2, \ldots$

$$\text{either } W_i \subset \overline{\text{span}}\{x_{ij}\}_{(i,j)\in I} \text{ or } W_i \subset \overline{\text{span}}\{x_{ij}\}_{(i,j)\in I^c}. \tag{51}$$

Then either I or I^c contains infinitely many W_i, without loss of generality we assume it is I. This means that for infinitely many i,

$$W_i \subset \overline{\text{span}}\{x_{ij}\}_{(i,j)\in I}. \tag{52}$$

Since $W_i \subset W_{i+1}$ for all i,

$$\cup_{i=1}^{\infty} W_i \subset \overline{\text{span}}\{x_{ij}\}_{(i,j)\in I}, \tag{53}$$

and so the closure of the right hand set is ℓ_2. This shows our family of vectors have complement property and hence do phase retrieval on ℓ_2. □

Theorem 6.2. *Let P_n be the orthogonal projection of ℓ_2 onto $E_n = \text{span}\{e_i\}_{i=1}^{n}$. There is a set of vectors $Y = \{y_{ni}\}_{n=1,i=1}^{\infty,\ \infty}$ that does not do phase retrieval on ℓ_2, but $X = \{x_{ni}\}_{n=1,i=1}^{\infty,\ \infty} = \{P_n y_{ni}\}_{n=1,i=1}^{\infty,\ \infty}$ does phase retrieval in ℓ_2. Moreover, finite subsets of X do phase retrieval on E_n for every n.*

Proof. For each $n \in \mathbb{N}$, let X_n be a finite set of vectors $\{x_{ni}\}_{i\in I_n}$ contained in E_n that does phase retrieval in E_n. For example consider a full spark set in E_n embedded in ℓ_2 by adding zero to all other entries. We know that $X = \{x_{ni}\}_{n=1,i\in I_n}^{\infty}$ does phase retrieval in ℓ_2. It is sufficient to show that for each n and i, there exists y_{ni}, with $P_n y_{ni} = x_{ni}$, such that the y_{ni}

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is contained in a fixed hyperplane for all n, i. Let w be the vector with infinitely many non-zero coordinates. For each n, x_{ni} has finite support contained in the first n coordinates, for all $i \in I_n$. Then there is $j > n$ such that $w(j) \neq 0$. Define $y_{ni} = x_{ni} - \frac{\langle x_{ni}, w\rangle}{w(j)} e_j$, for $i \in I_n$. It follows that $\langle y_{ni}, w\rangle = 0$, and hence $y_{ni} \subset w^{\perp}$ for all n, i. This completes the proof. □

In the following, we will show how to create a new phase retrieval set by translating the vectors of the original one in the same direction. First, we will need a lemma.

Lemma 6.3. *If $\{x_i\}_{i=1}^{\infty}$ is Bessel in ℓ_2, then for every $v \in \ell_2$,*

$$\lim_{i\to\infty} \langle v, x_i\rangle = 0. \tag{54}$$

Proof. Given a vector v, we have

$$\sum_{i=1}^{\infty} |\langle v, x_i\rangle|^2 < \infty, \tag{55}$$

hence $\lim_{i\to\infty} |\langle v, x_i\rangle| = 0$. □

Remark 6.4. Note that if any $\{x_i\}_{i=1}^{\infty}$ does phase retrieval, then

$$\left\{ \frac{1}{\|x_i\| 2^i} x_i \right\}_{i=1}^{\infty} \tag{56}$$

is Bessel and also does phase retrieval.

Theorem 6.5. *Assume $\{x_i\}_{i=1}^{\infty}$ is a Bessel sequence in ℓ_2 and does phase retrieval. Then for every $v \in \ell_2$, $\{x_i + v\}_{i=1}^{\infty}$ does phase retrieval.*

Proof. Assume

$$|\langle x, x_i + v\rangle| = |\langle y, x_i + v\rangle|, \text{ for all } i = 1, 2, \dots. \tag{57}$$

Let

$$I = \{i : \langle x, x_i + v\rangle = \langle y, x_i + v\rangle\}. \tag{58}$$

Then either $|I|$ or $|I^c|$ is infinite. By the complement property, either $\{x_i\}_{i\in I}$ or $\{x_i\}_{i\in I^c}$ spans the space. Without loss of generality, assume $\{x_i\}_{i\in I}$ spans ℓ_2. Now,

$$\langle x, x_i + v\rangle = \langle y, x_i + v\rangle, \text{ for all } i \in I, \tag{59}$$

and so

$$\langle x - y, x_i\rangle = \langle y - x, v\rangle, \text{ for all } i \in I.$$

By Lemma 6.3,

$$\langle y - x, v\rangle = 0 = \langle x - y, x_i\rangle, \text{ for all } i \in I. \tag{60}$$

It follows that $x - y = 0$. □

Remark 6.6. Note that $\{x_i + v\}_{i=1}^{\infty}$ is not Bessel. But we can scale it to be Bessel and it still does phase retrieval.

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Corollary 6.7. *We can perturb a family doing phase retrieval in ℓ_2 as long as we perturb all vectors in the same direction.*

Corollary 6.8. *Given $\{x_i\}_{i=1}^{\infty}$ which is Bessel and does phase retrieval in ℓ_2, for any x_j the family $\{x_i - x_j\}_{i=1}^{\infty}$ does phase retrieval.*

Proof. Let $v = -x_j$ and note that it follows that $\{x_i + v\}_{i=1}^{\infty}$ does phase retrieval by Theorem 6.5. □

Note repeating the argument in the previous corollary, it is possible to "delete" a finite number of vectors by translating the system and scaling the set so they are Bessel.

Proposition 6.9. *There is a family of vectors in ℓ_2 doing phase retrieval where each of the vectors has all non-zero coordinates with respect to the unit vectors.*

Proof. Let $\{x_i\}_{i=1}^{\infty}$ do phase retrieval. Let $\{e_i\}_{i=1}^{\infty}$ be the unit vectors. For any $j = 1, 2, \ldots$ the family $\{x_i(j)\}_{i=1}^{\infty}$ is a countable set of real numbers so choose a real number $a_j \neq -w_i(j)$ and $0 < a_j < \frac{1}{2^j}$ for all $i = 1, 2, \ldots$. Let $v = (a_1, a_2, \ldots)$. Then $\{x_i + v\}_{i=1}^{\infty}$ does phase retrieval and each vector has all non-zero coordinates. □

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Department of Mathematics, University of Missouri, Columbia, MO 65211-4100
E-mail address: sb8k6@missouri.edu, Casazzap@missouri.edu
E-mail address: chengdesai@yahoo.com, terraformthedreamscape@gmail.com
E-mail address: tinmizzou@gmail.com

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